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# NONLINEAR INERTIA-GRAVITY WAVE-MODE INTERACTIONS IN THREE DIMENSIONAL ROTATING STRATIFIED FLOWS*

MARK REMMEL†, JAI SUKHATME‡, AND LESLIE M. SMITH§

*Dedicated to the sixtieth birthday of Professor Andrew Majda*

**Abstract.** We investigate the nonlinear dynamics of inertia-gravity (IG) wave modes in three-dimensional (3D) rotating stratified fluids. Starting from the rotating Boussinesq equations, we derive a reduced partial differential equation system, the GGG model, consisting of only wave-mode interactions. We note that this subsystem conserves energy and is not restricted to resonant wave-mode interactions. In principle, comparing this model to the full rotating Boussinesq system allows us to gauge the importance of wave-vortical-wave vs. wave-wave-wave interactions in determining the transfer and distribution of wave-mode energy. As in many atmosphere-ocean phenomena we work in a skewed aspect ratio domain $H/L$ ($H$ and $L$ are the vertical and horizontal lengths) with $Fr = Ro < 1$ such that $Bu = 1$, where $Fr$, $Ro$, and $Bu$ are the Froude, Rossby, and Burger numbers, respectively. Our focus is on the equilibration of wave-mode energy and its spectral scaling under the influence of random large-scale ($k_f$) forcing. We present results from two sets of parameters: (i) $Fr = Ro \approx 0.05$, $H/L = 1/5$, and (ii) $Fr = Ro \approx 0.1$, $H/L = 1/3$. As anticipated from prior work, when forcing is applied to all modes with equal weight, with $Fr = Ro \approx 0.05$ and $H/L = 1/5$, the wave-mode energy of the full system equilibrates and its spectrum scales as a power-law that lies between $k^{-1}$ and $k^{-5/3}$ for $k_f < k < k_d$, where $k_d$ is the dissipation scale. For the same parameters, when forcing is restricted to only wave modes the wave-mode energy fails to equilibrate in both the full system as well as the GGG subsystem at the resolutions we can achieve. This clearly demonstrates the importance of the vortical mode (by facilitating wave-vortical-wave interactions) in determining the wave-mode energy in the rotating Boussinesq system. Proceeding to the second set of simulations, i.e., for the larger $Fr = Ro \approx 0.1$ in a less skewed aspect ratio domain with $H/L = 1/3$, we observe that the energy of the GGG subsystem equilibrates and is resolution independent. Furthermore, the full system with forcing restricted to wave modes also equilibrates and both yield identical power-law scaling of wave-mode energy spectra. Thus it is clear that the wave-wave-wave interactions play a role in the overall dynamics at moderate $Ro$, $Fr$ and aspect ratios. From a practical standpoint these results highlight the difficulty in properly resolving wave-mode interactions when simulating realistic geophysical phenomena.

**Key words.** Geophysical fluid dynamics, rotating-stratified turbulence, partial differential equations.

**AMS subject classifications.** 86A10, 76B15, 35Q35.

## 1. Introduction

Inertia gravity (IG) waves, resulting from the rotating and stratified nature of geophysical fluids, play an important role in atmosphere-ocean dynamics [1]. A broad overview of their properties with relevance to the ocean can be found in Garrett & Munk [2], and a recent review focussing on the observational characterization of the oceanic wave-field can be found in Polzin and Lvov [3]. The general dynamics of these waves are reviewed by Sommeria & Staquet [4]. Furthermore, Wunsch & Ferrari [5]

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†Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706, USA (remmel@math.wisc.edu).

‡Indian Institute of Tropical Meteorology, Pashan, Pune 41108, India and Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706, USA (sukhatme@math.wisc.edu).

§Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706 and Engineering Physics Department, University of Wisconsin-Madison, Madison, WI 53706, USA (lsmith@math.wisc.edu).