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MINIMAX VARIATIONAL PRINCIPLE FOR STEADY BALANCED SOLUTIONS OF THE ROTATING SHALLOW WATER EQUATIONS*

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Dedicated to the sixtieth birthday of Professor Andrew Majda

Abstract. Two well-known variational principles for geophysical flows are combined into a single minimax principle that characterizes distinguished steady solutions of the rotating shallow water (RSW) equations. On the one hand, in the limit of small Rossby number ϵ, in which the dynamics becomes quasi-geostrophic and closes terms of the potential vorticity field Q, steady coherent states are characterized as minimizers of (generalized) enstrophy $\mathcal{A}$ at a given value of total energy $\mathcal{H}$. On the other hand, for small amplitude motions at finite ϵ balanced states resulting from geostrophic adjustment are characterized as minimizers of the total energy $\mathcal{H}$ subject to a given potential vorticity Q. Moreover, the organization into a coherent state through potential vorticity mixing occurs on a slow time scale relative to the fast time scale of adjustment through inertia-gravity wave radiation. These two complementary principles suggest a variational characterization of steady balanced states for the RSW equations at finite ϵ. Namely, the functional $\mathcal{A}+\theta\mathcal{H}$, where $\theta<0$ is a parameter, is first maximized over all RSW fields with given Q, and then minimized over all Q. Any such minimax critical point of $\mathcal{A}+\theta\mathcal{H}$ is an exact steady solution of the RSW equations, which represents a physically relevant equilibrium state at finite Rossby number. This minimax principle is implemented numerically for zonal shear flows, and branches of solutions are computed to first-order in ϵ. The results quantify the breakdown of quasi-geostrophy and the asymmetry between cyclonic and anticyclonic structures. In addition, the $O(\epsilon)$-correction is computed for a model of the zonally-averaged winds in Jupiter's weather layer.

Key words. Rotating shallow water model, steady flow, variational principle, potential vorticity, geostrophic adjustment.

AMS subject classifications. 76U05, 76B55.

1. Introduction

The Rotating Shallow Water (RSW) equations constitute the simplest equations in geophysical fluid dynamics that govern both slow vortical motions and fast inertia-gravity waves. While they greatly simplify the vertical structure and thermodynamics of a stratified fluid, they provide a physically realistic model of key phenomena described by the full primitive equations [20, 29, 35]. In the limit of small Rossby number, the RSW equations become the quasi-geostrophic (QG) equations, which close in terms of the potential vorticity (PV) field and filter the inertia-gravity waves. In this limit there is an extensive modern theory of nonlinear coherent structures — organized flows, such as shear layers or vortices, that emerge and persist within geostrophic turbulence. A comprehensive presentation of this theory and its applications is given in the monograph by Majda and Wang [25]. The most complete part of this theory characterizes the statistical equilibrium states of quasi-geostrophic flows. Using the conservation properties of the PV transport equation as a basis, equilibrium states are realized as maximizers of the entropy of PV mixing subject to constraints on total energy and circulation. Such constrained entropy maximizing states are distinguished steady flows in two senses. First, they are most probable states with respect to fine-grained PV fluctuations; second, they are nonlinearly stable states with respect to coarse-grained perturbations [15, 16]. This modern equilibrium statistical

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