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FINITE-TIME BLOW-UP OF L^∞-WEAK SOLUTIONS OF AN AGGREGATION EQUATION*

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Dedicated to Andrew Majda on the occasion of his 60^{th} birthday

Abstract. We consider the aggregation equation $u_t+\nabla\cdot[(\nabla K)*u)u]=0$ with nonnegative initial data in $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ for $n\geq 2$. We assume that K is rotationally invariant, nonnegative, decaying at infinity, with at worst a Lipschitz point at the origin. We prove existence, uniqueness, and continuation of solutions. Finite time blow-up (in the L^∞ norm) of solutions is proved when the kernel has precisely a Lipschitz point at the origin.

Key words. Biological aggregation, 2D vorticity equation, finite-time blow-up, non-local PDE.

AMS subject classifications. 35A07, 35B60, 35D05, 35Q35, 35R05.

1. Introduction

In this paper we consider the evolution equation

$$u_t+\nabla\cdot[(\nabla K)*u)u]=0 \tag{1.1}$$

with nonnegative initial data in $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ for $n\geq 2$. We prove existence, uniqueness, and continuation of solutions. Finite time blow-up (in the L^∞ norm) of solutions is proved when the kernel has precisely a Lipschitz point at the origin. Equation (1.1) arises in the study of animal aggregations [11, 20, 22, 23] and also certain problems in materials science [13, 14]. In the case of animal aggregations, u represents population density, while in materials applications u typically represents a particle density. The case $K(x)=e^{-|x|}$, which we focus on here, arises in both the biological and materials literature [3, 13, 14, 20, 23].

By differentiation, (1.1) can be written as

$$\frac{\partial u}{\partial t}+\vec{v}\cdot\nabla u=-(\Delta K*u)u \quad\text{and}\quad \vec{v}=\nabla K*u. \tag{1.2}$$

This demonstrates that (1.1) is an advection-reaction equation; the solution u is amplified along characteristics by the nonlocal operator $(-\Delta K*u)u$. Problems such as (1.1), in which a quantity is transported by a vector field obtained by applying a nonlocal operator to that quantity, are known as active scalar problems [8]. Active scalar problems are common in fluid dynamics and have been used as model problems for vortex stretching in the 3-D Euler equations. One source of interest in vortex stretching is its intimate connection with the regularity of solutions of the incompressible Euler equations; it was proven in [2] that smooth solutions to the Euler equations develop singularities only if the vorticity becomes infinite in a certain sense. According to the Euler equations, the vorticity $\vec{\omega}$ satisfies

$$\frac{\partial \vec{\omega}}{\partial t}+\vec{v}\cdot\nabla\vec{\omega}=\vec{\omega}\cdot\nabla\vec{v} \quad\text{and}\quad \vec{v}=\vec{K_3}*\vec{\omega} \tag{1.3}$$

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