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GAUSSIAN BEAMS SUMMATION FOR THE WAVE EQUATION IN A CONVEX DOMAIN*

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Abstract. We consider the scalar wave equation in a bounded convex domain of $\mathbb{R}^n$. The boundary condition is of Dirichlet or Neumann type and the initial conditions have a compact support in the considered domain. We construct a family of approximate high frequency solutions by a Gaussian beams summation. We give a rigorous justification of the asymptotics in the sense of an energy estimate and show that the error can be reduced to any arbitrary power of ε, which is the high frequency parameter.

Key words. Wave equation, high frequency solutions, Gaussian beams summation, reflection at the boundary.

AMS subject classifications. 35L05, 35L20, 81S30, 41A60.

1. Introduction

In this paper, our aim is to provide asymptotic solutions, in a sense to be made more precise later, to the following initial-boundary value problem (IBVP) for the wave equation

$$\begin{cases} Pu_\varepsilon = \partial_t^2 u_\varepsilon - \partial_x \cdot (c^2(x)\partial_x u_\varepsilon) = 0 \text{ in } [0,T]\times\Omega, \\ u_\varepsilon|_{t=0} = u_\varepsilon^I, \partial_t u_\varepsilon|_{t=0} = v_\varepsilon^I \text{ in } \Omega, \\ Bu_\varepsilon = 0 \text{ in } [0,T]\times\partial\Omega, \end{cases} \tag{1.1}$$

where B is a Dirichlet or Neumann type boundary operator.

Above, $T>0$ is fixed, and Ω is a bounded domain of $\mathbb{R}^n$, with $n=2$ or $n=3$ for important applications to acoustics or elastodynamics problems.

We assume the boundary $\partial\Omega$ is $\mathcal{C}^\infty$ and the domain is convex for the bicharacteristic curves of P, see more precisely Assumption B1 below. Furthermore, the coefficient c is assumed to be in $\mathcal{C}^\infty(\bar{\Omega})$, though this assumption may be substantially relaxed.

Our initial data will depend on a small parameter $\varepsilon>0$, playing the role of a small wavelength, and our main objective is to study the high frequency limit, corresponding to $\varepsilon\to 0$, i.e., the construction of high frequency solutions. Moreover, we shall assume that u_ε^I, v_ε^I are

A1. uniformly bounded respectively in $H^1(\Omega)$ and $L^2(\Omega)$,

A2. uniformly supported in a fixed compact set $K\subset\Omega$.

The search for such approximate solutions and related notions of parametrices for the wave equation and similar equations has been an intensive area of research. A

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