COMMUN. MATH. SCI.
Vol. 7, No. 3, pp. 741–753

A NEW MEDIAN FORMULA WITH APPLICATIONS TO PDE BASED DENOISING*

YINGYING LI† AND STANLEY OSHER‡

Abstract. We develop a simple algorithm for finding the minimizer of the function $E(x)=\sum_{i=1}^{n} w_i|x-a_i|+F(x)$, when the w_i are nonnegative and F is strictly convex. If F is also differentiable and F' is bijective, we obtain an explicit formula in terms of a median. This enables us to obtain approximate solutions to certain important variational problems arising in image denoising. We also present a generalization with $E(x)=J(x)+F(x)$ for $J(x)$ a convex piecewise differentiable function with a finite number of nondifferentiable points.

Key words. Convex optimization, ℓ_1 minimization, TV denoising, Bregman iterative method.

AMS subject classifications. 46N10, 94A08.

1. Introduction

Given $a_1,\dots,a_n\in\mathbb{R}$, it is well-known that

$$\min_{x\in\mathbb{R}}\sum_{i=1}^{n}|x-a_i|^{\alpha}=\begin{cases}\text{mean}(a_i) & \text{if } \alpha=2\\ \text{median}(a_i) & \text{if } \alpha=1\\ \text{mode}(a_i) & \text{if } \alpha=0.\end{cases}$$

More generally, the very early work of Barral [14] investigated

$$\min_{x\in\mathbb{R}}\sum_{i=1}^{n}w_i|x-a_i|^{\alpha},\quad w_i\geq 0,\alpha=0,1,2,\infty.$$

Also, [15, 16] developed local M-estimator filters based on such minimizers.

This work was inspired by two variational problems arising in image research. One is soft wavelet thresholding [2, 8] or basis pursuit [3] arising in compressed sensing. The other is the Rudin-Osher-Fatemi (ROF) model [1] of TV-based image denoising and generalizations. The first involves reducing $E(x)=\sum_{i=1}^{n}w_i|x-a_i|+F(x)$ to its simplest terms: the scalar problem

$$\min_{x\in\mathbb{R}}E(x)=|x|+\lambda(x-f)^2,\quad \lambda>0. \tag{1.1}$$

The solution to (1.1) is obtained from a simple formula: $x_{opt}=\text{shrink}(f,\frac{1}{2\lambda})$, where the shrink operator can be found in figure 1.1

It turns out that

$$\text{shrink}(f,\tfrac{1}{2\lambda})=\text{median}\{f-\tfrac{1}{2\lambda},0,f+\tfrac{1}{2\lambda}\}. \tag{1.2}$$

This seems (surprisingly) to be a new result. We generalize it below.

*Received: May 24, 2009; accepted (in revised version): July 20, 2009. Communicated by Martin Burger.

†UCLA Department of Mathematics, Los Angeles, CA, 90095-1555 (yingyingli@math.ucla.edu).
‡UCLA Department of Mathematics, Los Angeles, CA, 90095-1555 (sjo@math.ucla.edu). This research was supported by ONR Grant N00014710810 and the Department of Defense.