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SHARP CONSTANT IN NONLOCAL INEQUALITY AND ITS APPLICATIONS TO NONLOCAL SCHRÖDINGER EQUATION WITH HARMONIC POTENTIAL*

JIANQING CHEN†, BOLING GUO‡, AND YONGQIAN HAN§

Abstract. This paper contains two parts. In the first part, we derive a variant of Gagliardo-Nirenberg interpolation inequality involving nonlocal nonlinearity and determine its best (smallest) constant. In the second part, we study two applications of this inequality and its best constant. In the first application, we use this best constant to establish a sharp criterion for the global existence and blow-up of solutions of the inhomogeneous Schrödinger equation with harmonic potential and nonlocal nonlinearity

$$i\varphi_t = -\triangle\varphi + |x|^2\varphi - \varphi|\varphi|^{p-2}\int \frac{|\varphi(y)|^p}{|x-y|^\alpha}dy$$

in the critical case $p=2+(2-\alpha)/N$. The result indicates that the existence of blow-up solutions not only depends on the mass of the initial data but also on the profile of the initial data. In the second application, we use this best constant to prove that when $2+(2-\alpha)/N<p<(2N-\alpha)/(N-2)$, the solutions exist globally in time for one class of initial data whose norm can be as large as one wants.

Key words. Nonlocal interpolation inequality, sharp constant, global solutions, blow-up, nonlocal Schrödinger equation.

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1. Introduction

This paper is concerned with the nonlinear Schrödinger equation with harmonic potential

$$i\varphi_t + \triangle\varphi - |x|^2\varphi + F(\varphi) = 0, \quad x\in\mathbb{R}^N, \quad t\geq 0, \tag{1.1}$$

where $\varphi := \varphi(x,t)\colon\ \mathbb{R}^N\times\mathbb{R}_+\to\mathbb{C}$ is a complex-valued function and $F(\varphi)$ is a nonlinearity (possibly nonlocal) satisfying suitable assumptions given later.

Equation (1.1) models a lot of physical phenomena. For example it is known as the Gross-Pitaevskii (GP) equation in the context of Bose-Einstein condensates (BEC) with parabolic traps. In fact, assuming a highly anistropic trap, Kivshar et al [17] derived the GP-equation equation (1.1) with $F(\varphi)=\varphi|\varphi|^2$ as a model equation for the macroscopic dynamics of cooled atoms confined in a three dimensional parabolic potential created by a magnetic trap. Deconinck et al derive a three dimensional Schrödinger equation with potential and nonlocal nonlinearity; see [10, Equ. (5)].

Equation (1.1) with nonlocal nonlinearity $F(\varphi)$ has also appeared in other applications. For example, Kurth derived in [18] that Schrödinger equation with harmonic potential and nonlocal nonlinearity can be used to describe average pulse propagation

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†School of Mathematics and Computer Science, Fujian Normal University, Cangshan Campus, Fuzhou 350007, P.R. China (chenjq_iapcm@126.com; jqchen@fjnu.edu.cn).

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‡Institute of Applied Physics and Computational Mathematics, PO Box 8009, Beijing 100088, P.R. China.

§Institute of Applied Physics and Computational Mathematics, PO Box 8009, Beijing 100088, P.R. China.