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A LEVEL SET METHOD FOR THE COMPUTATION OF MULTIVALUED SOLUTIONS TO QUASI-LINEAR HYPERBOLIC PDES AND HAMILTON-JACOBI EQUATIONS*

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Abstract. We develop a level set method for the computation of multivalued solutions to quasi-linear hyperbolic partial differential equations and the gradient flow of the Hamilton-Jacobi equations in any number of space dimensions. We use the classic idea of Courant and Hilbert to define the solution of the quasi-linear hyperbolic PDEs or the gradient of the solution to the Hamilton-Jacobi equations as zero level sets of level set functions. Then the evolution equations for the level set functions satisfy *linear Liouville equations* defined in the "phase" space, unfolding the singularities and preventing the numerical capturing of the viscosity solution. This provides a computational framework for the computation of multivalued geometric solutions to general quasilinear PDEs. By using the local level set method the cost of each time update for this method is $O(N^d \log N)$ for a d dimensional problem, where N is the number of grid points in each dimension.

Key words. Hyperbolic PDEs, Hamilton-Jacobi equations, level set method, multivalued solution, Liouville equation

1. Introduction

Many physical problems require the computation of multivalued solutions. Such problems arise in the computation of dispersive waves [44, 27, 15], geometric optics [11, 33] , semiclassical linear and nonlinear Schrödinger equations [16, 25, 24, 39], multiple arrival in tomography and seismic migration [14, 41, 43], electron beam modulation in vacuum electronic devices (such as Klystron) [21, 28], etc. Although these problems are often, in the continuum limit, described by nonlinear hyperbolic partial differential equations or Hamilton-Jacobi equations, the classical entropy or viscosity solutions are not adequate in describing the post singularity behavior. The underlying physics is often defined by Hamiltonian systems which are non dissipative, often dispersive, and whose solution can be superimposed in the phase space. Direct numerical discretizations of these nonlinear PDEs often yield viscosity solutions which are irreversible, thus violating underlying physical principles, such as superposition.

There are two classes of methods used to compute multivalued solutions. A classical technique is ray tracing, which is a Lagrangian method that solves a set of ordinary differential equations in order to trace the wavefronts. This method is easy to implement, but may encounter difficulties in spatial resolution when points that are close initially may diverge at later times. This results in the loss of numerical accuracy unless a regridding and/or interpolation procedure is implemented from time to time. Another class involves Eulerian methods, which solve partial differential equations on a fixed grid, see e.g. [2],[3]. The new difficulty lies in the fact that a solution becomes multivalued in the physical space beyond singularity (caustic) formation.

In recent years, there has developed a growing interest in the computation of multivalued solutions of these nonlinear PDEs using Eulerian methods. Methods based in physical space often use moment closure of the classical Liouville equations.

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