Commun. Math. Sci.
Vol. 7, No. 1, pp. 129–141

GLOBAL SOLUTIONS FOR THE ONE DIMENSIONAL WATER-BAG MODEL*

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Abstract. In this paper we study a special type of solution for the one dimensional Vlasov-Maxwell equations. We assume that initially the particle density is constant on its support in the phase space and we are looking for solutions where the particle density has the same property at any time $t>0$. More precisely, for each x the support of the density is assumed to be an interval $[p^-,p^+]$ with end-points varying in space and time. Here we analyze the case of weak and strong solutions for the effective equations satisfied by the end-points and the electric field (water-bag model) in the relativistic setting.

Key words. Vlasov-Maxwell equations, water-bag model, conservation laws.

AMS subject classifications. 35A05, 78A35, 82D10.

1. Introduction

The Vlasov-Maxwell system governs the evolution of an ensemble of charged particles subject to electromagnetic fields created by themselves and possibly external sources in which collisions are typically neglected. Given f, the density number of charged particles at time $t\in\mathbb{R}_+$, position $x\in\mathbb{R}^3$ and momentum $p\in\mathbb{R}^3$, the dynamics of the particles is described by the Vlasov equation

$$\partial_t f+v(p)\cdot\nabla_x f+q(E(t,x)+v(p)\wedge B(t,x))\cdot\nabla_p f=0,\ (t,x,p)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3, \quad (1.1)$$

where the electromagnetic field (E,B) is defined in a self-consistent way by the Maxwell equations

$$\partial_t E-c_0^2\mathrm{curl}_x B=-\frac{j(t,x)}{\varepsilon_0},\ j(t,x)=q\int_{\mathbb{R}^3}v(p)f(t,x,p)\,dp,\ (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \quad (1.2)$$

$$\partial_t B+\mathrm{curl}_x E=0,\ (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \quad (1.3)$$

$$\mathrm{div}_x E=\frac{\rho(t,x)}{\varepsilon_0},\ \rho(t,x)=q\int_{\mathbb{R}^3}f(t,x,p)\,dp,\ \mathrm{div}_x B=0,\ (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \quad (1.4)$$

where q,m are the charge and the mass of the particles, ε_0 is the electric permittivity of the vacuum, and $v(p)$ is the relativistic velocity associated to the momentum p

$$v(p)=\frac{p}{m}\left(1+\frac{|p|^2}{m^2c_0^2}\right)^{-\frac{1}{2}},$$

where c_0 is the speed of light in a vacuum. Suitable initial conditions for the particle density and the electromagnetic field have to be prescribed satisfying certain compatibility conditions. The existence of global weak solutions was obtained in [10] and the existence of strong solutions has been investigated by different approaches in

*Received: April 30, 2008; accepted (in revised version): November 4, 2008. Communicated by François Bouchut.

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