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STABILITY AND TOTAL VARIATION ESTIMATES ON GENERAL SCALAR BALANCE LAWS*

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Abstract. Consider the general scalar balance law $\partial_t u + \mathrm{Div} f(t,x,u) = F(t,x,u)$ in several space dimensions. The aim of this paper is to estimate the dependence of its solutions on the flow f and on the source F. To this aim, a bound on the total variation in the space variables of the solution is obtained. This result is then applied to obtain well posedness and stability estimates for a balance law with a non local source.

Key words. Multi-dimensional scalar conservation laws, Kružkov entropy solutions.

AMS subject classifications. 35L65.

1. Introduction

The Cauchy problem for a scalar balance law in N space dimensions

$$\begin{cases} \partial_t u + \mathrm{Div} f(t,x,u) = F(t,x,u) & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u(0,x) = u_o(x) & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

is well known to admit a unique weak entropy solution, as proved in the classical result by Kružkov [12, Thm. 5]. The same paper also provides the basic stability estimate on the dependence of solutions from the initial data, see [12, Thm. 1]. In the same setting established in [12], we provide here an estimate on the dependence of the solutions to (1.1) on the flow f and the source F, and recover the known estimate on the dependence from the initial datum u_o. A key intermediate result is a bound on the total variation of the solution to (1.1), which we provide in Theorem 2.5.

In the case of a conservation law, i.e., where $F = 0$, and where the flow f is independent of t and x, the dependence of the solution on f was already considered in [3], where other results were also presented. In this case, the TV bound is obvious, since $\mathrm{TV}\big(u(t)\big) \le \mathrm{TV}(u_o)$. The estimate provided by Theorem 2.5 slightly improves the analogous result in [3, Thm. 3.1] (that was already known, see [6, 16]), which reads (for a suitable absolute constant C)

$$\big\| u(t) - v(t) \big\|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} \le \| u_o - v_o \|_{\mathbf{L^1}(\mathbb{R}^N;\mathbb{R})} + C\, \mathrm{TV}(u_o)\, \mathbf{Lip}(f-g)\, t.$$

Our result, given by Theorem 2.6, reduces to this inequality when f and g are not dependent on t and x and $F = G = 0$, but with $C = 1$.

A flow also dependent on x was considered in [4, 9], though in the special case $f(x,u) = l(x)\, g(u)$, but with a source term containing a possibly degenerate parabolic

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