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CONVERGENCE OF A TIME DISCRETIZATION FOR A CLASS OF NON-NEWTONIAN FLUID FLOW*

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Dedicated to Professor Rolf D. Grigorieff on the occasion of his 70th birthday

Abstract. The equation describing the non-stationary flow of an incompressible non-Newtonian fluid is approximated by the fully- and semi-implicit two-step backward differentiation formula (BDF). The stress tensor is assumed to be of p-structure such that the usual coercivity, growth, and monotonicity condition is fulfilled. Convergence of a piecewise polynomial prolongation of the discrete solution towards an exact weak solution is shown for the case $p \geq 1+2d/(d+2)$, where d denotes the spatial dimension.

Key words. Non-Newtonian fluid flow, weak solution, time discretization, two-step scheme, convergence, monotone operator.

AMS subject classifications. 76A05, 65M12, 47H05, 35Q35.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d \in \{2,3\}$) be an open bounded set with boundary $\partial\Omega$ of class $\mathcal{C}^{0,1}$, and let $(0,T)$ be the time interval under consideration. The flow of an incompressible fluid can be described by the initial-boundary value problem

$$\begin{aligned} &\partial_t u - \nabla\cdot\sigma + (u\cdot\nabla)u + \nabla p = f, \quad \nabla\cdot u = 0 \quad \text{in } \Omega\times(0,T), \\ &u = 0 \quad \text{on } \partial\Omega\times(0,T), \quad u(\cdot,0) = u_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$ denotes the velocity field with the prescribed initial velocity $u_0 = u_0(x)$, $p = p(x,t)$ is the pressure, and $f = f(x,t)$ is an external force per unit mass. The symmetric stress tensor $\sigma = \sigma(e)$ is assumed to be a continuous function in the symmetrized velocity gradient $e(u) = \big(\nabla u + (\nabla u)^T\big)/2$ and is assumed to fulfill the following structural assumption: *There are numbers $p > 1$, μ, $c > 0$ such that for all $y, z \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$*

$$\sigma(z)\cdot z \geq \mu|z|^p, \quad |\sigma(z)| \leq c(1+|z|)^{p-1}, \quad (\sigma(z)-\sigma(y))\cdot(z-y) \geq 0. \tag{1.2}$$

Note that $y\cdot z := \sum_{i,j=1}^d y_{ij}z_{ij}$ for $y,z \in \mathbb{R}^{d\times d}$, $|z| := (z\cdot z)^{1/2}$, and $y\cdot z = y\cdot z^T$ if $y^T = y$. Typical examples are

- the Stokes law with $\sigma(z) \sim z$ that leads (with $p = q = 2$) to the Navier-Stokes equation,
- power-law fluids with $\sigma(z) \sim |z|^{p-2}z$ that describe so-called shear thickening if $p > 2$ and shear thinning if $1 < p < 2$, respectively, and
- variants of the power-law such as $\sigma(z) \sim (1+|z|^2)^{(p-2)/2}z$.

A discussion of the non-Newtonian model can be found in standard monographs as e.g. [4, 7, 27]. Other than the model above, other descriptions of complex fluid flow that do not follow Newton's linear relation between stress and strain have been studied. Examples are viscoelastic fluid flow such as the Oldroyd model (see [14, 22,

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