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THE DERIVATIVES OF ASIAN CALL OPTION PRICES *

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Abstract. The distribution of time integrals of geometric Brownian motion is not well understood. To price an Asian option and to obtain measures of its dependence on the parameters of time, strike price, and underlying market price, it is essential to have the distribution of time integral of geometric Brownian motion and it is also required to have a way to manipulate its distribution. We present integral forms for key quantities in the price of Asian option and its derivatives (*delta, gamma, theta, and vega*). For example for any $a>0$, $\mathbb{E}\left[(A_t-a)^+\right]=t-a+a^2\,\mathbb{E}\left[(a+A_t)^{-1}\exp\left(\frac{2M_t}{a+A_t}-\frac{2}{a}\right)\right]$, where $A_t=\int_0^t \exp(B_s-s/2)\,ds$ and $M_t=\exp(B_t-t/2)$.

Key words. Asian option, derivatives of option prices, geometric Brownian motion, time integral

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1. Introduction

The payoff of an Asian option depends on the (geometric or arithmetic) average of prices of a given risky asset over the pre-specified time interval. Under the Black-Scholes framework, one assumes that the price process $\{S_t,\ t\geq 0\}$ of the risky asset follows

$$dS_t=\mu S_t dt+\sigma S_t dB_t, \qquad S_0>0$$

where μ and σ are given constants and $\{B_t,\ t\geq 0\}$ is a standard one dimensional Brownian motion. In this setting, it is easy to understand the geometric average. If $0\leq t_1<t_2$ then

$$\sqrt{S_{t_1}\cdot S_{t_2}}=S_0\exp\left[\ \left(\sigma\sqrt{t_2+3t_1}\right)\cdot\mathcal{N}+(\mu-\frac{\sigma^2}{2})(t_1+t_2)\ \right]$$

in distribution, where $\mathcal{N}$ is a standard normal random variable. On the other hand, the distribution of an arithmetic average process is not well understood. A continuous version of the arithmetic average is a time integral of a price process. Using the Inverse Lapalce Transformation, Yor [15] proved many interesting identities related to the distribution of geometric Brownian motion, which gives us deeper understanding of functions of geometric Brownian motion and useful information about their time integrals. More detailed research for the relation between the time integral and an Asian option was considered in [6]. Using the joint density of $\int_0^t \exp(B_s)dW_s,\exp(B_t)$, where B_t, W_t are independent Brownian motions given in [2], the moment generating function of the time integral process was computed in [11]. The method of changing measures was considered to analyze the properties of the time integral process, (see [7, 12, 13, 14].) In [5] the very useful time reversing property is used to analyze the time integral process. Dufresne also provided a certain form for the density function of the time integral of geometric Brownian motion. However the author pointed out

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