Commun. Math. Sci.
Vol. 6, No. 1, pp. 235–256

A BEALE-KATO-MAJDA BREAKDOWN CRITERION FOR AN OLDROYD-B FLUID IN THE CREEPING FLOW REGIME*

RAZ KUPFERMAN†, CLAUDE MANGOUBI‡, AND EDRISS S. TITI§

Abstract. We derive a criterion for the breakdown of solutions to the Oldroyd-B model in $\mathbb{R}^3$ in the limit of zero Reynolds number (creeping flow). If the initial stress field is in the Sobolev space $H^m(\mathbb{R}^3)$, $m>5/2$, then either a unique solution exists within this space indefinitely, or, at the time where the solution breaks down, the time integral of the L^∞-norm of the stress tensor must diverge. This result is analogous to the celebrated Beale-Kato-Majda breakdown criterion for the inviscid Euler equations of incompressible fluids.

Key words. Oldroyd-B model, local-in-time existence, Beale-Kato-Majda

AMS subject classifications. 35B35, 35Q35

1. Introduction

The Oldroyd-B model is a classical model for dilute solutions of polymers suspended in a viscous incompressible solvent [1]. Although it suffers, as a physical model derived from microscopic dynamics, from numerous shortcomings (e.g., polymers are allowed to stretch indefinitely), it is often considered as a prototypical model for viscoelastic fluids, and has therefore been the focus of both analytical and numerical studies.

At present, there is no global-in-time existence theory for the Oldroyd-B model. The notable difference between the Oldroyd-B model and its Newtonian counterpart, the incompressible Navier-Stokes equations, is that in the viscoelastic case, global-in-time existence has not even been established in two dimensions nor in the creeping flow regime, i.e., when the momentum equation is a Stokes system. The reason for this difference can be understood by observing structural similarities between the inertialess Oldroyd-B model and the Euler equations (in three dimensions), or the 2D quasi-geostrophic flow equations (in two dimensions) [2].

Since the early 1970s, numerical simulations of the Oldroyd-B model (as well as other viscoelastic models) have been infested by stability and accuracy problems that arise at frustratingly low values of the elasticity parameter (the Weissenberg number) [3, 4]. While some of these difficulties have been elucidated [5], it is to a large extent still a mystery why computations break down in the low-Reynolds-high-Weissenberg regime. As is often the case in such situations, numerical data are by themselves not sufficient to explain the reasons for this breakdown. In the absence of a well-posedness theory, it is not even clear in what spaces solutions have to be sought. Thus, the development of such a theory is of major importance for both theoretical and practical purposes.

This situation is analogous to that of incompressible Newtonian fluids at high Reynolds number, where global-in-time existence has not yet been established. For a

*Received: August 14, 2007; accepted (in revised version): December 4, 2007. Communicated by Weinan E.

†Institute of Mathematics, The Hebrew University, Jerusalem 91904 Israel.

‡Institute of Mathematics, The Hebrew University, Jerusalem 91904 Israel and CERMICS, École Nationale des Ponts et Chaussées, 77455 Marne la Vallée, France.

§Faculty of Mathematics and Computer Science, The Weizmann Institute of Science, Rehovot 76100 Israel and Department of Mathematics and Department of Mechanical and Aerospace Engineering, University of California, Irvine CA 92697-3875.