A BEALE-KATO-MAJDA BREAKDOWN CRITERION FOR AN OLDROYD-B FLUID IN THE CREEPING FLOW REGIME∗

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Abstract. We derive a criterion for the breakdown of solutions to the Oldroyd-B model in $\mathbb{R}^3$ in the limit of zero Reynolds number (creeping flow). If the initial stress field is in the Sobolev space $H^m(\mathbb{R}^3)$, $m > 5/2$, then either a unique solution exists within this space indefinitely, or, at the time where the solution breaks down, the time integral of the $L^\infty$-norm of the stress tensor must diverge. This result is analogous to the celebrated Beale-Kato-Majda breakdown criterion for the inviscid Euler equations of incompressible fluids.

Key words. Oldroyd-B model, local-in-time existence, Beale-Kato-Majda

AMS subject classifications. 35B35, 35Q35

1. Introduction

The Oldroyd-B model is a classical model for dilute solutions of polymers suspended in a viscous incompressible solvent [1]. Although it suffers, as a physical model derived from microscopic dynamics, from numerous shortcomings (e.g., polymers are allowed to stretch indefinitely), it is often considered as a prototypical model for viscoelastic fluids, and has therefore been the focus of both analytical and numerical studies.

At present, there is no global-in-time existence theory for the Oldroyd-B model. The notable difference between the Oldroyd-B model and its Newtonian counterpart, the incompressible Navier-Stokes equations, is that in the viscoelastic case, global-in-time existence has not even been established in two dimensions nor in the creeping flow regime, i.e., when the momentum equation is a Stokes system. The reason for this difference can be understood by observing structural similarities between the inertialess Oldroyd-B model and the Euler equations (in three dimensions), or the 2D quasi-geostrophic flow equations (in two dimensions) [2].

Since the early 1970s, numerical simulations of the Oldroyd-B model (as well as other viscoelastic models) have been infested by stability and accuracy problems that arise at frustratingly low values of the elasticity parameter (the Weissenberg number) [3, 4]. While some of these difficulties have been elucidated [5], it is to a large extent still a mystery why computations break down in the low-Reynolds-high-Weissenberg regime. As is often the case in such situations, numerical data are by themselves not sufficient to explain the reasons for this breakdown. In the absence of a well-posedness theory, it is not even clear in what spaces solutions have to be sought. Thus, the development of such a theory is of major importance for both theoretical and practical purposes.

This situation is analogous to that of incompressible Newtonian fluids at high Reynolds number, where global-in-time existence has not yet been established. For a

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