COMMUN. MATH. SCI.
Vol. 6, No. 1, pp. 217–234

AVALANCHES IN NETWORKS OF WEAKLY COUPLED PHASE-SHIFTING ROTATORS*

A.YU. SHAHVERDIAN† AND A.V. APKARIAN‡

Abstract. The networks of phase-shifting rotators interacting through exchange of weak δ-kicks are considered. Such a rotator consists of a particle rotating on a circle which at some discrete moments receives some δ-kicks. We assume that the kicks are not of mechanical character: they change a particle's position but not the rate. A comparison of the rotator networks with the BTW model of self-organized criticality, Burridge-Knopoff's model in seismicity, Herz-Hopfield neural networks, and the Turing-Smale system in biological cells is presented. This work studies the avalanches in rotator networks — the events when a number of rotators almost simultaneously hit some threshold levels. The asymptotic relations linking distribution of avalanches with the architecture of a network are proved. The equivalence of two well-known power-law conjectures, in lattice models of statistical physics and in interacting threshold systems, is established.

Key words. dynamical networks, self-organized criticality, avalanches

PACS: 05.45.Xt.

1. Introduction

Let us explain our abstract and describe the networks of threshold microsystems considered in this work; the formal definitions are given in the following sections. Let $E_N=[1,2,\dots,N]$ be the sequence of $N\geq 2$ integers and $\bar{\omega}=(\omega_1,\dots,\omega_N)\in\mathbb{R}^N$ be a vector with non-zero components. Let $\mathcal{R}=\{R_1,\dots,R_N\}$ be a set of N rotators R_i (some of them may coincide): the R_i consists of a circle C_i and a particle P_i rotating on it with constant angular rate $|\omega_i|$; rotation occurs clockwise or anti-clockwise depending on whether ω_i is positive or negative. Let to every $i\in E_N$ a label $\rho_i\in C_i$ as well as a nonempty set of indices $\sigma_i\subset E_N$, $i\notin\sigma_i$ be prescribed: ρ_i is understood as some threshold position for the motion of P_i and $\bar{\sigma}=(\sigma_1,\dots,\sigma_N)$ assigns the neighborhood in the network $\mathcal{R}$: R_i interacts with R_j iff $j\in\sigma_i$. The interaction is as follows: each time when P_i hits the level ρ_i, each P_j whose index j is in σ_i instantaneously rotates (shifts) along its orbit on C_j by a given small angle (the intensity of interaction). The following is required: a shifting of a particle does not affect its rate (it remains the same before and after interaction) and the moments when the particle hits its threshold are separated by a positive constant. This work studies the avalanches in such rotator networks — the events when a number of rotators in the network almost simultaneously attain their corresponding thresholds.

We claim that if in a system of almost identical rotators the interaction of small intensity is included, then provided certain extent in couplings there arises a dynamics which is complicated. The situation is reminiscent of Turing's statement [20, 22] in biological cells: Turing-Smale's theorem says that when put into conditions allowing certain interaction, the system of initially inert cells begins to exhibit nontrivial oscillatory dynamics. It can be said that we study the Turing-Smale statement adapted to the framework of self-organized criticality (SOC), a theory resulting from numerous studies in the BTW model (a description is given below) which deals with large

*Received: October 28, 2007; accepted (in revised version): December 30, 2007. Communicated by David Cai.

†Corresponding author: Institute for Informatics and Automation Problems, Armenian National Academy of Sciences, Paruir Sevak Str., 1, Yerevan, 0014, Armenia (svrdn@yerphi.am).

‡Department of Physiology, Northwestern University, The Medical School, 303 East Chicago Ave., Tarry Bldg. 5-703, Chicago, Illinois, 60611 USA (a-apkarian@northwestern.edu).