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BREAKDOWN OF HOMOGENIZATION FOR THE RANDOM HAMILTON-JACOBI EQUATIONS*

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Abstract. We study the homogenization of Lagrangian functionals of Hamilton-Jacobi equations (HJ) with quadratic nonlinearity and unbounded stationary ergodic random potential in R^d, $d\geq 1$. We show that homogenization holds if and only if the potential is bounded from above. When the potential is unbounded from above, homogenization breaks down, due to the almost sure growth of the running maxima of the random potential. If the unbounded randomness appears in the advection term, homogenization may or may not hold depending on the structure of the flow field. In (compressible) unbounded gradient flows, homogenization holds in spite of the unboundedness. In (incompressible) unbounded shear flows, homogenization breaks down again due to unbounded running maxima of the flows. Results for random advection follow from a transformation of the problem to that of HJ with random potential. Analogous effective behavior is present for front speeds in reaction-diffusion-advection equations with unbounded random advection, and may have broader implications for wave propagation in random media.

Key words. random Lagrangian functionals, homogenization, divergence, extreme behavior

AMS subject classifications. 76M50, 60H05, 76M35

1. Introduction

Stochastic Hamilton-Jacobi equations (HJ) appear as useful prototype models for wave propagation and growth phenomena in random media, see [3, 14, 15] among others. The effective large space-time behavior can be captured by studying the homogenization limit ($\epsilon\to 0$) of the associated random Lagrangian functional:

$$S^{\epsilon}(x,t,\omega)=\inf_{\xi\in A}\int_0^t L(\xi(s)/\epsilon,\xi'(s),\omega)\,ds, \tag{1.1}$$

where $A=\{\xi\in W^{1,\infty}([0,t];R^d):\xi(0)=0,\xi(t)=x\}$; $L=L(x,q,\omega)$, $x,q\in R^d$ $(d\geq 1)$, is the Lagrangian (Legendre transform) of a convex random Hamiltonian $H(x,p,\omega)$ to be specified. The earlier works [11, 10] showed that if the Lagrangian (Hamiltonian) is convex in q (p) with superlinear growth in large $|q|$ ($|p|$), and uniformly bounded in x, then S^{ϵ} converges almost surely to a deterministic function, $S^*(x,t)=t\mu^*(\frac{x}{t})$, where $\mu^*=\mu^*(q)$ is a convex function with superlinear growth in q. The Legendre transform of μ^* gives the effective Hamiltonian $H^*=H^*(p)$ which defines the homogenized HJ equation $u_t+H^*(\nabla_x u)=0$. For affine initial data $u(x,0)=p\cdot x$, the solution is a propagating front $u(x,t)=p\cdot x-H^*(p)t$, where $H^*(p)$ is the effective front speed in unit direction p.

In this paper, we consider **Hamiltonians or Lagrangians which are unbounded in** x; for example, the x dependence can be an unbounded Gaussian process. We are interested in the existence of the homogenization limit (1.1). It turns out that the existence of the limit is quite subtle because of the unbounded randomness. When the Hamiltonian is of the form: $H(x,p,\omega)=|p|^2/2+V(x,\omega)$, $p,x\in R^d$,

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