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FAST COMMUNICATION

MODERATE DISPERSION IN CONSERVATION LAWS WITH CONVEX FLUXES*

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Abstract. We consider the weakly dissipative and weakly dispersive Burgers-Hopf-Korteweg-de-Vries equation with the diffusion coefficient ε and the dispersion rate δ in the range $\delta/\varepsilon \to 0$. We study the traveling wave connecting $u(-\infty)=1$ to $u(+\infty)=0$ and show that it converges strongly to the entropic shock profile as $\varepsilon,\delta \to 0$.

Key words. traveling waves, moderate dispersion, Korteweg de Vries equation, entropy solutions, dissipative-dispersive scalar conservation laws

AMS subject classifications. 35B25, 35Q53, 35L67

1. Introduction

This paper is concerned with hyperbolic conservation laws approximated by a weakly dissipative and weakly dispersive equation; that is, we are interested in the limit as $\varepsilon \to 0$, $\delta \to 0$ of the solutions of the following problem

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(A(u)) = \varepsilon \frac{\partial^2 u}{\partial x^2} - \delta \frac{\partial^3 u}{\partial x^3},$$
$$u(t,0) = u^0(x). \tag{1.1}$$

It is well known that when the parameter ε vanishes too fast compared to δ, the dispersive effects dominate and produce oscillations. In that case the (weak) limit is not a weak solution to the conservation law with $\varepsilon = \delta = 0$ ([9]). Therefore several authors have considered the *weak dispersion* case $\delta = \alpha\varepsilon^2$, with $0 < \alpha < \infty$ fixed, showing that the solutions of (1.1) converge strongly ([7, 8, 11]) to weak solution to the equation with $\varepsilon = \delta = 0$. Such limits may lead to non-entropic solutions [1, 6, 7, 10, 11] for non-convex fluxes $A(u)$. However, for strictly convex fluxes and in this weak dispersion regime $\delta = \alpha\varepsilon^2$ with $\alpha > 0$ fixed, one expects that the (strong) limits always satisfy the family of Kruzkov entropies; this is proved in [10] for instance for traveling waves. For general solutions to the initial value problem in the weak dispersion regime, it is easy to prove that the square entropy satisfies the entropy inequality but there is no direct derivation of the full family of entropy inequalities. However, an indirect argument due to R. DiPerna indicates that for convex fluxes a single entropy inequality implies all the others; the original argument uses the BV regularity of solutions (such a bound is not available here) and this regularity assumption has been removed recently in [3, 4]. But we are not aware of any related result in the moderate dispersion regime.

The purpose of this paper is to further investigate the case of convex fluxes, on

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