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LARGE PARAMETER BEHAVIOR OF EQUILIBRIUM MEASURES*

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Abstract. We study the equilibrium measure for a logarithmic potential in the presence of an external field $V_*(\xi)+t\ p(\xi)$, where t is a parameter, $V_*(\xi)$ is a smooth function and $p(\xi)$ a monic polynomial. When $p(\xi)$ is of an odd degree, the equilibrium measure is shown to be supported on a single interval as $|t|$ is sufficiently large. When $p(\xi)$ is of an even degree, the equilibrium measure is supported on two disjoint intervals as t is negatively large; it is supported on a single interval for convex $p(x)$ as t is positively large and is likely to be supported on multiple disjoint intervals for non-convex $p(x)$.

The support of the equilibrium measure shrinks to isolated points as $|t|\to+\infty$ in all the cases that we consider. For sufficiently large $|t|$, each topological component of the support contains a local minimizing point of the external field $V_*(\xi)+t\ p(\xi)$; a "potential well" phenomenon.

Key words. Equilibrium measure, zero dispersion limit, random matrix

AMS subject classifications. 15A52, 35Q15, 35Q53, 35Q58

1. Introduction

In this paper, we study the following minimization problem with constraints

$$\underset{\{\psi\geq 0,\int\psi d\xi=1\}}{\text{Minimize}}\left[-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\log|\xi-\eta|\psi(\xi)\psi(\eta)d\xi d\eta+\int_{-\infty}^{+\infty}V(\xi)\psi(\xi)d\xi\right]\ . \tag{1.1}$$

The external field $V(\xi)$ is a C^∞ function that satisfies

$$\lim_{\xi\to\pm\infty}\frac{V(\xi)}{log(1+\xi^2)}=+\infty\ . \tag{1.2}$$

Under this condition, the existence and uniqueness of the minimizer for (1.1) has been established [20]. The measure $\psi(\xi)d\xi$, where $\psi(\xi)$ is the minimizer of (1.1), is called the equilibrium measure under the external field $V(\xi)$.

Equilibrium measures find applications in many branches of mathematical sciences. It is used to describe the partition function of the Hermitian one-matrix model in random matrix theory of statistical physics [2, 18]. It is also intrinsically connected to the free energy of the Yang-Mills theory [9]. Finally, it plays an important role in orthogonal polynomials and approximation theory [20].

Although its importance to physics and approximation theory ([8, 20]) is well known, the minimization problem (1.1) is not well understood. The minimizer is explicitly known only for a few cases where the external fields $V(\xi)$ are the simplest polynomials [1, 6, 20]. It is therefore desirable to study the minimization problem for much more general C^∞ external fields.

Our method to solve the minimization problem (1.1) is the same as one [12] we used to solve a similar minimization problem for the zero dispersion limit of the KdV equation

$$u_t+6uu_x+\epsilon^2u_{xxx}=0\quad\text{with } u(x,0;\epsilon)=u_0(x)\ ,$$

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