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The Isoperimetric Problem for Pinwheel Tilings

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Abstract: In aperiodic "pinwheel" tilings of the plane there exist unions of tiles with ratio (area)/(perimeter)^2 arbitrarily close to that of a circle. Such approximate circles can be constructed with arbitrary center and any sufficiently large radius. The existence of such circles follows from the metric on pinwheel space being almost Euclidean at large distances; if P and Q are points separated by large Euclidean distance R, then the shortest path along tile edges from P to Q has length $R + o(R)$.

I. Introduction and Statement of Results

The classic isoperimetric problem in the plane, which asks for the curve of least length enclosing some fixed area, has stimulated much important mathematics. One generalization which has developed within geometric measure theory treats spaces less symmetric than the Euclidean plane, such as spaces representing the structure of crystals. Due to the periodic arrangement of their atoms such structures are, on a macroscopic scale, invariant under translations but not rotations. This has easily observed consequences for crystals; for quartz or table salt one can literally see an optimal polyhedral shape, a shape which solves a version of the isoperimetric problem that can be described as follows. There is a "cost function" $f(\vec{n})$ associated with variable normal directions $\vec{n}$ of planes in space. (Physically, $f(\vec{n})$ is the energy per unit area needed to separate a crystal into two parts along a plane with normal $\vec{n}$.) The problem is to imagine integrating f over each possible surface enclosing a region of fixed volume V, and then to find the surface which minimizes this integral. In 1901 Wulff gave a simple construction for such optimal surfaces (see [Wul, Tay]) which is still used to produce the polyhedral "Wulff shapes" for common crystals.

In this paper we generalize this isoperimetric problem to geometries associated with quasicrystals. Quasicrystals are an exotic class of solids, usually metallic alloys,

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