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On Ulam–von Neumann Transformations

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Abstract: We define and study Ulam–von Neumann transformations which are certain interval mappings and conjugate to $q(x) = 1 - 2x^2$ on $[-1, 1]$. We use a singular metric on $[-1, 1]$ to study a Ulam–von Neumann transformation. This singular metric is universal in the sense that it does not depend on any particular mapping but only on the exponent of this mapping at its unique critical point. We give the smooth classification of Ulam–von Neumann transformations by their eigenvalues at periodic points and exponents and asymmetries.

Contents

1. Introduction

Ulam and von Neumann studied the chaotic behavior of the nonlinear self mapping $q(x) = 1 - 2x^2$ of the interval $[-1, 1]$ in 1947. They observed that $\rho_q = 1/(\pi\sqrt{1-x^2})$ is the density function of a unique absolutely continuous q-invariant measure (we only consider probability measures). In modern language, this observation shows that q is a chaotic dynamical system and follows from making the singular change of metric $dy = (2/\pi)(dx/\sqrt{1-x^2})$. If we let $y = h(x)$ be the corresponding change of coordinate and $\tilde{q} = h \circ q \circ h^{-1}$, then q becomes $\tilde{q}(y) = 1 - 2|y|$, a piecewise linear mapping with expansion rate 2 on $[-1, 1]$. The dynamics of $\tilde{q}$ is more easily understood.

Now consider a general self mapping f of $[-1, 1]$ whose graph looks like those in Fig. 1. Then f is topologically conjugate to q under a certain smoothness

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