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On Diagonalization in Map$(M, \mathbf{G})$

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Abstract: Motivated by some questions in the path integral approach to (topological) gauge theories, we are led to address the following question: given a smooth map from a manifold M to a compact group $\mathbf{G}$, is it possible to smoothly "diagonalize" it, i.e. conjugate it into a map to a maximal torus $\mathbf{T}$ of $\mathbf{G}$?

We analyze the local and global obstructions and give a complete solution to the problem for regular maps. We establish that these can always be smoothly diagonalized locally and that the obstructions to doing this globally are non-trivial Weyl group and torus bundles on M. We explain the relation of the obstructions to winding numbers of maps into $\mathbf{G}/\mathbf{T}$ and restrictions of the structure group of a principal $\mathbf{G}$ bundle to $\mathbf{T}$ and examine the behaviour of gauge fields under this diagonalization. We also discuss the complications that arise in the presence of non-trivial $\mathbf{G}$-bundles and for non-regular maps.

We use these results to justify a Weyl integral formula for functional integrals which, as a novel feature not seen in the finite-dimensional case, contains a summation over all those topological $\mathbf{T}$-sectors which arise as restrictions of a trivial principal $\mathbf{G}$ bundle and which was used previously to solve completely Yang–Mills theory and the G/G model in two dimensions.

1. Introduction

One of the most useful properties of a compact Lie group $\mathbf{G}$ is that its elements can be "diagonalized" or, more formally, conjugated into a fixed maximal torus $\mathbf{T} \subset \mathbf{G}$. In this paper we investigate to which extent this property continues to hold for spaces of (smooth) maps from a manifold M to a compact Lie group $\mathbf{G}$. Thus, given a smooth map $g : M \to \mathbf{G}$, the first thing one would like to know is if it can be written as

$$g(x) = h(x)\, t(x)\, h^{-1}(x)\,, \tag{1.1}$$

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