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A Lax Representation for the Vertex Operator and the Central Extension

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Abstract: Integrable hierarchies, viewed as isospectral deformations of an operator L may admit symmetries; they are time-dependent vector fields, transversal to and commuting with the hierarchy and forming an algebra. In this work, the commutation relations for the symmetries are shown to be based on a non-commutative Lie algebra splitting theorem. The symmetries, viewed as vector fields on L, are expressed in terms of a Lax pair.

This study introduces a "generating symmetry", a generating function for symmetries, both of the KP equation (continuous), and the two-dimensional Toda lattice (discrete), in terms of L and an operator M, introduced by Orlov and Schulman, such that $[L,M] = 1$. This "generating symmetry", acting on the wave function (or wave vector) lifts to a vertex operator *à la* Date-Jimbo-Kashiwara-Miwa, acting on the τ-function (or τ-vector). Lifting the algebra of symmetries, acting on wave functions, to an algebra of symmetries, acting on τ-functions, amounts to passing from an algebra to its central extension.

This provides a handy technology to find the constraints satisfied by various matrix integrals, arising in the context of $2d$-quantum gravity and moduli space topology. The point is to first prove the vanishing of symmetries at the Lax pair level, which usually turns out to be elementary and conceptual, and then use the lifting above to get the subalgebra of vanishing symmetries for the τ-function (or τ-vectors).

0. Introduction and Main Results

Most integrable equations are part of a hierarchy of equations, and can be viewed as isospectral deformations of a system of linear equations, with one or more sequences of scalar deformation parameters $t = (t_1, t_2, \ldots)$ as independent variables. They may come equipped with so-called *symmetries*, which are vector fields acting on the space of solutions of the hierarchy, which may explicitly depend on time, and which commute with the hierarchy, but not necessarily among themselves. Symmetries