Commun. Math. Phys. 171, 139 – 158 (1995)

Communications in Mathematical Physics

The Tangent Bundle of a Calabi-Yau Manifold – Deformations and Restriction to Rational Curves

D. Huybrechts

Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, 53225 Bonn, Germany; Institute for Advanced Study, School of Mathematics, Olden Lane, Princeton, 08540 NJ, USA

Received: 10 May 1994 / in revised form: 27 October 1994

Abstract: The tangent bundle $\mathscr{T}_X$ of a Calabi-Yau threefold X is the only known example of a stable bundle with non-trivial restriction to any rational curve on X. By deforming the direct sum of $\mathscr{T}_X$ and the trivial line bundle one can try to obtain new examples. We use algebro-geometric techniques to derive results in this direction. The relation to the finiteness of rational curves on Calabi-Yau threefolds is discussed.

0. Introduction

In [W] Witten posed the following question:

Can one deform the vector bundle $\mathscr{T}_X \oplus \mathscr{O}_X$ to a stable vector bundle whose restriction to any rational curve is nontrivial?

Here $\mathscr{T}_X$ is the tangent bundle of a Calabi-Yau threefold X and $\mathscr{O}_X$ is the trivial line bundle on it. He showed that such deformations are of significance in string theory (existence of flat directions in the superpotential). In fact, $(X, \mathscr{T}_X)$ seems to be the only known example for a pair (X, E) consisting of a Calabi-Yau manifold X and a stable vector bundle E with nontrivial restriction to any rational curve. A positive answer to the above question would provide an example with a rank four bundle whose Chern classes are those of X. This problem and certain generalizations of it were also formulated in problem 77 in Yau's recent problem list [Y].

This paper grew out of the attempt to understand the problem in algebro-geometric terms and to use the available techniques in deformation theory to derive some first results in special cases. In particular we prove:

- Let X be embedded as a hypersurface and assume that it can be deformed in the ambient space to another Calabi-Yau threefold X' not isomorphic to X with $X \cap X' \neq \emptyset$ (e.g. X is a complete intersection). Then $\mathscr{T}_X \oplus \mathscr{O}_X$ can be deformed to a stable bundle (1.3).
- For the generic quintic $X \subset \mathbf{P}_4$ there exists a stable deformation of $\mathscr{T}_X \oplus \mathscr{O}_X$ whose restriction to all lines, i.e. rational curves of degree one, is not trivial (3.3).