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The Geodesic Approximation for the Yang–Mills–Higgs Equations

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Abstract: In this paper we consider the dynamics of the monopole solutions of Yang–Mills–Higgs theory on Minkowski space. The monopoles are solutions of the Yang–Mills–Higgs equations on three dimensional Euclidean space. It is of interest to understand how they evolve in time when considered as solutions of the Yang–Mills–Higgs equations on Minkowski space-i.e. the time dependent equations. It was suggested by Manton that in certain situations the monopole dynamics could be understood in terms of geodesics with respect to a certain metric on the space of guage equivalence classes of monopoles-the moduli space. The metric is defined by taking the L^2 inner product of tangent vectors to this space. In this paper we will prove that Manton's approximation is indeed valid in the right circumstances, which correspond to the slow motion of monopoles. The metric on the moduli space of monopoles was analysed in a book by Atiyah and Hitchin, so together with the results of this paper a detailed and rigorous understanding of the low energy dynamics of monopoles in Yang–Mills–Higgs theory is obtained. The strategy of the proof is to develop asymptotic expansions using appropriate gauge conditions, and then to use energy estimates to prove their validity. For the case of monopoles to be considered here there is a technical obstacle to be overcome-when the equations are linearised about the monopole the continuous spectrum extends all the way to the origin. This is overcome by using a norm introduced by Taubes in a discussion of index theory for the Yang–Mills–Higgs functional.

1. Introduction

In this paper we will construct certain solutions of the Yang–Mills–Higgs equations on Minkowski space. To write these down let $(x_0,x_1,x_2,x_3) = (t,x_1,x_2,x_3)$ be coordinates on Minkowski space, then the dependent variables are an $su(2)$ valued one form called the *connection*:

$$A = A_0 dt + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$$

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