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Onsager's Conjecture on the Energy Conservation for Solutions of Euler's Equation

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Abstract: We give a simple proof of a result conjectured by Onsager [1] on energy conservation for weak solutions of Euler's equation.

In this note, we give a simple proof of a result conjectured by Onsager [1] on the energy conservation for weak solutions of the 3D incompressible Euler's equation. To avoid questions regarding boundaries, we will assume periodic boundary conditions with period box $D = [0,1]^3$. We will use the summation convention and frequently suppress the independent variable t for notational convenience. We use $B_p^{\alpha,q}$ to denote the Besov spaces.

Theorem. *Let* $u = (u_1, u_2, u_3) \in L^3([0,T], B_3^{\alpha,\infty}(D)) \cap C([0,T], L^2(D))$ *be a weak solution of the 3D incompressible Euler's equation*, i.e.

$$-\int_0^T\int_D u_j(x,t)\partial_t\psi_j(x,t)d^3x dt - \int_D u_j(x,0)\psi_j(x,0)d^3x$$

$$-\int_0^T\int_D \partial_i\psi_j(x,t)u_i(x,t)u_j(x,t)d^3x dt - \int_0^T\int_D \partial_i\psi_i(x,t)\,p(x,t)d^3x dt = 0 \tag{1}$$

for every test function $\psi = (\psi_1,\psi_2,\psi_3) \in C^\infty(D \times R^1)$ *with compact support. If* $\alpha > \frac{1}{3}$, *then we have*

$$\int_{R^3} |u(x,t)|^2 d^3x = \int_{R^3} |u(x,0)|^2 d^3x, \quad \textit{for } t \in [0,T)\,. \tag{2}$$

This is basically the content of Onsager's conjecture, except Onsager stated his conjecture in Hölder spaces rather than Besov spaces. Obviously the above theorem implies similar results in Hölder spaces. We state the results in the Besov spaces for two reasons: The first is that it gives the sharp result. The second is that the