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Averaging for Diffusions with Fine-Grained Boundaries

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Abstract: This article investigates the limiting behavior of a diffusion in a half space with a complicated boundary condition. The boundary condition implements a reflection condition everywhere except a number of small sets or "holes" that meet Dirichlet or mixed boundary conditions. Probabilistic methods associated with the Feynman-Kac formula are used to find the limiting behavior of the diffusion equations as the number of holes gets large and the size of each hole is reduced. With particular scaling homogenization occurs, and we see that the complicated boundary condition is replaced by a simple mixed boundary condition depending on the capacitance and distribution of the holes.

1. Problem Formulation

Our model problem investigates diffusions in the half space $\mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}^+$ with a fine-grained boundary condition. Our boundary implements Neumann boundary conditions, $\partial u/\partial x_3(x_1, x_2, 0) = 0$, everywhere on $\{x : x_3 = 0\}$ except at a number of small sets. These exceptional sets are scaled, rotated, and translated copies of a nominal set H. On these exceptional sets, Dirichlet conditions, $u(x_1, x_2, 0) = 0$, or mixed conditions, $\partial u/\partial x_3(x_1, x_2, 0) - \lambda u(x_1, x_2, 0) = 0$, occur. We refer to these exceptional sets as "holes," and the number of holes is scaled by N while the size of each hole is scaled by $1/N$. Figure 1 shows an example of a nominal hole H, and Fig. 2 shows a possible boundary configuration with $N = 5$. Our goal is to study averaging effects as $N \to \infty$.

The properties of these collections of small sets are first formally described.

Hole Assumption I. *Let $H \subset \mathfrak{R}^3$ be a bounded closed set contained in the plane*

$$P \equiv \{(x_1, x_2, x_3) : x_3 = 0\}$$

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