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Yang-Mills Fields which are not Self-Dual

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Abstract. The purpose of this paper is to prove the existence of a new family of non-self-dual finite-energy solutions to the Yang-Mills equations on Euclidean four-space, with $SU(2)$ as a gauge group. The approach is that of "equivariant geometry:" attention is restricted to a special class of fields, those that satisfy a certain kind of rotational symmetry, for which it is proved that (1) a solution to the Yang-Mills equations exists among them; and (2) no solution to the self-duality equations exists among them. The first assertion is proved by an application of the direct method of the calculus of variations (existence and regularity of minimizers), and the second assertion by studying the symmetry properties of the linearized self-duality equations. The same technique yields a new family of non-self-dual solutions on the complex projective plane.

Introduction

The Yang-Mills functional (or "energy") is defined on the space of connections on the principal $SU(2)$-bundle $\mathbb{R}^4 \times SU(2)$ on Euclidean four-space $\mathbb{R}^4$ by assigning to a connection A the L^2-norm of its curvature F_A

$$YM(A) := -\int_{\mathbb{R}^4} \operatorname{tr}(F_A \wedge * F_A),$$

where "$*$" is the Hodge duality operator on 2-forms on Euclidean $\mathbb{R}^4$. The corresponding variational equations are the Yang-Mills equations

$$d_A * F_A = 0, \tag{1}$$

where d_A is the covariant exterior derivative associated with the connection A. From the Bianchi identity, $d_A F_A = 0$, it follows that if A is (anti-)self-dual,

$$F_A = \pm * F_A \tag{2}$$

("$+$" for self-dual, "$-$" for anti-self-dual), then it satisfies the Yang-Mills equations.

We will be concerned in this paper only with *finite-energy* solutions to Eq. (1) and (2). An example of a nontrivial (i.e. with nonvanishing curvature) finite-