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A Quantum Field Theoretic Description of Linking Numbers and Their Generalization

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Abstract. It is shown that the linking number of two surfaces of dimension p and $n-p-1$ in an n dimensional manifold has a natural description in terms of the two-point function of a simple topological field theory. The two-point function of a more general theory (whose partition function yields the Ray–Singer torsion) provides a definition for a generalized linking number.

1. Introduction

The language of quantum field theory provides a natural framework for describing a variety of results in mathematics. These include the Donaldson invariants of four manifolds [1], the Floer cohomology groups of three manifolds [1], and the Jones polynomials of knot theory [2]. The key idea, is all cases, is to construct a field theory which is independent of a spacetime metric. There are thus no local excitations and the only observables are topological invariants.

The results mentioned above refer to the topology of low dimensional manifolds. Although in many respects this is the richest case to consider, it is of interest to note that there exists a quantum field theoretic description of topological invariants in higher dimensions as well. In fact, the first topological quantum field theory [3,4] was constructed (over ten years ago) to reproduce the Ray–Singer analytic torsion [5]. A basic topological invariant in higher dimensional manifolds is the linking number of a p and $n-p-1$ dimensional submanifold. We will show that this linking number arises naturally from the two-point function of a simple topological field theory. This theory can be viewed as a special case of the theory which yields the Ray–Singer torsion. We will also consider this extended theory and find that the two-point function yields a generalization of the linking number. This is one example of the new insight that the quantum field theoretic approach can bring to topology.

We begin by reviewing the definition of the linking number [6]. Let M be a compact, oriented n dimensional manifold (without boundary). Let U and V be nonintersecting oriented submanifolds of dimension p and $n-p-1$. We assume U and V are homologically trivial, i.e., they are the boundaries of