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Analyticity and Global Existence of Small Solutions to Some Nonlinear Schrödinger Equations

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Abstract. In this paper we will study the nonlinear Schrödinger equations:

$$i\partial_t u + \tfrac{1}{2}\Delta u = |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n_x,$$
$$u(0,x) = \phi(x), \quad x \in \mathbb{R}^n_x. \qquad (*)$$

It is shown that the solutions of $(*)$ exist and are analytic in space variables for $t \in \mathbb{R}\backslash\{0\}$ if $\phi(x)$ $(\in H^{2n+1,2}(\mathbb{R}^n_x))$ decay exponentially as $|x| \to \infty$ and $n \geqq 2$.

1. Introduction and Results

We consider the nonlinear Schrödinger equations

$$i\partial_t u + \tfrac{1}{2}\Delta u = |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n_x, \tag{1.1}$$

$$u(0,x) = \phi(x), \quad x \in \mathbb{R}^n_x. \tag{1.2}$$

There are many works on the global existence of solutions to (1.1)–(1.2) in appropriate Sobolev spaces (see [2–5, 8, 10–13], and references cited therein). Furthermore it is known that (1.1)–(1.2) have a smoothing property that the solutions become smooth for $t \neq 0$ even if their initial data are not smooth. More precisely, in [7] it was shown that all solutions of (1.1)–(1.2) are smooth for $t \neq 0$ provided that the initial data in $H^{[n/2]+1,2}(\mathbb{R}^n_x)$ decay sufficiently rapidly as $|x| \to \infty$. Our aim of this paper is twofold. One is to show that if the initial data ϕ are analytic and sufficiently small in an appropriate norm, then the solutions of (1.1)–(1.2) exist globally in time and are analytic in space variables. The other is to show that if the initial data ϕ in $H^{2n+1,2}(\mathbb{R}^n_x)$ decay exponentially as $|x| \to \infty$ and are sufficiently small in an appropriate norm, then the solutions of (1.1)–(1.2) exist globally in time and are analytic in space variables for $t \in \mathbb{R}\backslash\{0\}$.

We give notation and function spaces used in this paper.

Notation and Function Spaces. We let $L^p(\mathbb{R}^n_x) = \{f(x); f(x)$ is measurable on $\mathbb{R}^n_x$, $|f(x)|_{L^p(\mathbb{R}^n_x)} < \infty\}$, where $|f(x)|_{L^p(\mathbb{R}^n_x)} = \left(\int_{\mathbb{R}^n_x} |f(x)|^p dx\right)^{1/p}$ if $1 \leqq p < \infty$ and $|f(x)|_{L^\infty(\mathbb{R}^n_x)} =$