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Inverse Scattering Transform for the Time Dependent Schrödinger Equation with Applications to the KPI Equation

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Abstract. For the direct-inverse scattering transform of the time dependent Schrödinger equation, rigorous results are obtained based on an operator-triangular-factorization approach. By viewing the equation as a first order operator equation, similar results as for the first order $n \times n$ matrix system are obtained. The nonlocal Riemann–Hilbert problem for inverse scattering is shown to have solution.

1. Introduction

We study in this paper the direct-inverse scattering problem for the $1+1$ time dependent Schrödinger equation:

$$i\psi_y + \psi_{xx} = -u\psi, \tag{1.1}$$

for real or complex potentials u. This problem, besides being of independent physical interest, is connected with the Cauchy problem of the Kadomtsev–Petviashvili (I) (KPI) equation

$$(u_t + 6uu_x + u_{xxx})_x = 3y_{yy}. \tag{1.2}$$

It has been formally studied by Zakharov and Manakov [Z–M], [M], Fokas and Ablowitz [F–A]. For the rigorous theory, certain estimates for the direct scattering transform have been obtained by Segur [S]. The work [Z–M], [M] contains important ideas such as triangular factorization of operators and the derivation of the positivity of the $\bar{\partial}$-scattering data $I + \mathscr{F}$ (see (3.20)) from the unitarity of the physical scattering data $I + \mathscr{S}$ (see (3.5)), while the work [F–A] gives a different approach for deriving $\mathscr{F}$ and constructs for the first time the lump solutions which are two dimensional soliton solutions. However, for the inverse scattering transform, even on the formal level, a satisfactory treatment has not yet been obtained (see [S] for the comments on [M]).

Our approach is based on viewing Eq. (1.1) as a first order operator equation in y (2.31). For the direct-inverse scattering problem, Eq. (1.1) behaves much more like a first order system than a one dimensional Schrödinger equation