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The Rotation Set of a Homeomorphism of the Annulus is Closed

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Abstract. We show that the rotation set for any orientation preserving, boundary component preserving homeomorphism of the annulus is closed. In particular, if the homeomorphism is area preserving, then the rotation set is a closed interval.

Introduction

In this paper we show that for any orientation preserving, boundary component preserving homeomorphism $f:\mathbf{A}\to\mathbf{A}$ of the closed annulus, and for any lift $\tilde{f}:\tilde{\mathbf{A}}\to\tilde{\mathbf{A}}$ to the universal cover, the rotation set $R(\tilde{f})$ is closed. This generalizes work of Aubry [Au–D] and of Mather [Ma] who showed (among other things) that if f is an area preserving twist map of the annulus, then $R(\tilde{f})$ is a closed interval.

We identify $\tilde{\mathbf{A}}$ with $\mathbf{R}\times[0,1]$ and let $p_1:\tilde{\mathbf{A}}\to\mathbf{R}$ be the projection onto the first coordinate. The covering translation for $\tilde{\mathbf{A}}$ is $T(x,y)=(x+1,y)$. For each $x\in\mathbf{A}$, choose a lift $\tilde{x}\in\tilde{\mathbf{A}}$ and consider $\lim\limits_{n\to\infty}(p_1\tilde{f}^n(\tilde{x})-p_1(\tilde{x}))/n$. When this limit exists, it is called the *$\tilde{f}$-rotation number of* x and denoted $\varrho(\tilde{f},x)$; as the notation indicates, it is independent of the choice of $\tilde{x}$. Although $\varrho(\tilde{f},x)$ need not be defined for all x, it is defined μ-a.e. for every f-invariant measure μ.

Theorem 0.1. *If $f:\mathbf{A}\to\mathbf{A}$ is an orientation preserving, boundary component preserving homeomorphism and $\tilde{f}:\tilde{\mathbf{A}}\to\tilde{\mathbf{A}}$ is any lift, then:*

1. *The rotation set $R(\tilde{f})=\bigcup\varrho(\tilde{f},x)$ is a closed set, where the union is taken over the domain of ϱ.*
2. *For each $r\in R(\tilde{f})$, there is an f-invariant measure μ_r such that $\varrho(\tilde{f},x)=r$ for μ_r-a.e. $x\in\mathbf{A}$.*
3. *With the exception of at most a discrete set of values r in $R(\tilde{f})$, there is a compact invariant set Q_r such that $\varrho(\tilde{f},x)=r$ for all $x\in Q_r$; if r is rational then Q_r exists and is realized by a periodic orbit.*