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Quantal Problems with Partial Algebraization of the Spectrum

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Abstract. We discuss a new class of spectral problems discovered recently which occupies an intermediate position between the exactly-solvable problems (e.g., harmonic oscillator) and all others. The problems belonging to this class are distinguished by the fact that a part of the eigenvalues and eigenfunctions can be found algebraically, but not the whole spectrum. The reason explaining the existence of the quasi-exactly-solvable problems is a hidden dynamical symmetry present in the hamiltonian. For one-dimensional motion this hidden symmetry is $SL(2, R)$. It is shown that other groups lead to a partial algebraization in multidimensional quantal problems. In particular, $SL(2, R) \times SL(2, R)$, $SO(3)$ and $SL(3, R)$ are relevant to two-dimensional motion inducing a class of quasi-exactly-solvable two-dimensional hamiltonians. Typically they correspond to systems in a curved space, but sometimes the curvature turns out to be zero. Graded algebras open the possibility of constructing quasi-exactly-solvable hamiltonians acting on multicomponent wave functions. For example, with a (non-minimal) superextension of $SL(2, R)$ we get a hamiltonian describing the motion of a spinor particle.

1. Introduction

Recent investigations [1–7] of the spectral problem of Schrödinger type have led to a surprising finding: a new type of problem has been discovered in which a part of the spectrum can be found by purely algebraic methods (quasi-exactly-solvable problems according to terminology of [4]). In the present paper we develop the approach proposed in [2, 3, 6]. The main idea lying in the basis of this approach is the existence of a hidden dynamical symmetry inherent to the hamiltonians of quasi-exactly-solvable type.

Any hamiltonian H can obviously be represented as an infinite-dimensional hermitian matrix,

$$H \Rightarrow \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n} & \dots \\ h_{21} & h_{22} & \dots & h_{2n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ h_{n1} & h_{n2} & \dots & h_{nn} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} . \tag{1}$$