Complete Ricci-Flat Kähler Manifolds of Infinite Topological Type

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Abstract. We display an infinite dimensional family of complete Ricci-flat Kähler manifolds of complex dimension 2, for which the second homology is infinitely generated. These are obtained from the Gibbons-Hawking Ansatz [2] by using infinitely many, sparsely distributed centers.

Introduction

In [2], Gibbons and Hawking construct families of complete Ricci-flat Kähler metrics on a class of non-compact 4-manifolds N_k. The metrics are asymptotically locally Euclidean in the sense that $\partial N_k \approx S^3/\mathbb{Z}_k$, and the metrics approach, at infinity, the locally Euclidean metric on the cone $C(S^3/\mathbb{Z}_k)$. Another description of these metrics was given by Hitchin [3]. Further examples, with boundary a spherical space form S^3/Γ, $\Gamma \subset SU(2)$, and a characterization of these metrics (Torelli theorem) among asymptotically locally Euclidean metrics were obtained by Kronheimer [5, 6].

In this paper, we show that one may also obtain complete Ricci-flat Kähler metrics corresponding to the case "$k = \infty$" of the Gibbons-Hawking metrics. These metrics are no longer asymptotically locally Euclidean, or of finite action, and are carried by a 4-manifold whose 2nd homology is infinitely generated. It is only recently (7) that examples of complete metrics of non-negative Ricci curvature have been exhibited on manifolds of infinite topological type.

The example shows that a complex 2-manifold supporting a complete Ricci-flat Kähler metric need not be the complement of a divisor in a compact complex surface since the homology of such a complement is certainly finitely generated. This indicates that a conjecture of Yau [8, 9] concerning the existence of such compactifications is not true without some strengthening of the hypothesis.

These metrics also provide the first example for which the moduli space of complete Ricci-flat metrics on a given manifold is infinite dimensional.