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Exactly Soluble Diffeomorphism Invariant Theories

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Abstract. A class of diffeomorphism invariant theories is described for which the Hilbert space of quantum states can be explicitly constructed. These theories can be formulated in any dimension and include Witten's solution to $2+1$ dimensional gravity as a special case. Higher dimensional generalizations exist which start with an action similar to the Einstein action in n dimensions. Many of these theories do not involve a spacetime metric and provide examples of topological quantum field theories. One is a version of Yang-Mills theory in which the only quantum states on $S^3 \times R$ are the θ vacua. Finally it is shown that the three dimensional Chern-Simons theory (which Witten has shown is intimately connected with knot theory) arises naturally from a four dimensional topological gauge theory.

1. Introduction

Gravity in three dimensions is often considered trivial since the field equation $R_{\mu\nu}=0$ implies that the spacetime is flat. However this does not mean that there is only one classical solution. On a manifold of topology $\Sigma \times R$, where Σ is a compact two manifold of genus $g>1$, there is a $12g-12$ dimensional space of flat metrics [1]. Since the solution space is finite dimensional, the theory is analogous to ordinary particle mechanics and the quantization should be straightforward.

However if one writes the three dimensional Einstein action in canonical form using the usual canonical variables (the spatial metric and its conjugate momentum) then one finds that the constraints associated with diffeomorphism invariance are

$$p_{ij}p^{ij}-(p^i_i)^2-R=0, \tag{1.1}$$

$$D_i p^{ij}=0, \tag{1.2}$$

where D_i and R are the covariant derivative and scalar curvature of the spatial metric. These are essentially identical in form to those of the four dimensional

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