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The Inverse Backscattering Problem in Three Dimensions

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Abstract. This article is a study of the mapping from a potential $q(x)$ on $\mathbf{R}^3$ to the backscattering amplitude associated with the Hamiltonian $-\Delta + q(x)$. The backscattering amplitude is the restriction of the scattering amplitude $a(\theta, \omega, k), (\theta, \omega, k) \in S^2 \times S^2 \times \mathbb{R}_+$, to $a(\theta, -\theta, k)$. We show that in suitable (complex) Banach spaces the map from $q(x)$ to $a(x/|x|, -x/|x|, |x|)$ is usually a local diffeomorphism. Hence in contrast to the overdetermined problem of recovering q from the full scattering amplitude the inverse backscattering problem is well posed.

This article is a study of the mapping from a potential on $\mathbf{R}^3$ to its quantum mechanical scattering amplitude. The scattering amplitude associated with a potential $q(x)$ can be described as follows. One assumes that for each $k > 0$ and each $\omega \in S^2$,

$$(-\Delta + q - k^2)u = 0$$

has a unique solution of the form $\exp(ik\omega \cdot x) + v(x, \omega, k)$ such that $v = \lim_{\varepsilon \downarrow 0} v_\varepsilon$, where v_ε is the square-integrable solution of

$$-\Delta v_\varepsilon + q v_\varepsilon - (k + i\varepsilon)^2 v_\varepsilon = -e^{ik\omega \cdot x} q. \tag{I.1}$$

Much work has been devoted to showing that, under general hypotheses on $q, v(x, \omega, k)$ exists and is unique (see Agmon [1], and the references given there). When $q \in C_0^\infty(\mathbf{R}^3)$ and hence $\Delta v + k^2 v \in C_0^\infty(\mathbf{R}^3)$, it is an elementary consequence of (I.1) that

$$v(x) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{e^{ik|x-y|}}{|x-y|} (\Delta + k^2) v(y) dy, \quad \text{and hence}$$

$$v(x) = \left(\frac{e^{ik|x|}}{4\pi |x|} \right) (a(x/|x|, \omega, k) + O(|x|^{-1})) \tag{I.2}$$

as $|x| \to \infty$. The function $a(\theta, \omega, k)$ on $S^2 \times S^2 \times \mathbf{R}_+$ is known as the scattering amplitude. If we replace functions in (I.2) by their Fourier transforms, we have