Commun. Math. Phys. 120, 233–248 (1988)

Communications in
Mathematical
Physics

On the Invariant Mass Conjecture in General Relativity

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Abstract. An asymptotic symmetries theorem is proved under certain hypotheses on the behaviour of the metric at spatial infinity. This implies that the Einstein–von Freud–ADM mass can be invariantly assigned to an asymptotically flat four dimensional end of an asymptotically empty solution of Einstein equations if the metric is a no-radiation metric or if the end is defined in terms of a collection of boost-type domains.

1. Introduction

One of the still unsolved classical problems in general relativity is to establish well-posedness of at least one of the existing definitions of energy-momentum at spatial infinity of an asymptotically flat space-time. Whatever the framework used to define energy-momentum [Ei, We, ADM, Ge, AH, Som, AD] the problems arising are closely related to the one which appears when one tries to define it via the so-called von Freud superpotential [vF]:

$$p_\mu = \lim_{R\to\infty} \frac{3}{16\pi} \int_{\substack{r(x)=R \\ x^0=\mathrm{const}}} \delta^{\alpha\beta\gamma}_{\lambda\nu\mu}\eta^{\lambda\rho}\eta_{\gamma\sigma}g^{\nu\sigma}_{,\rho}dS_{\alpha\beta}, r(x) = \{\textstyle\sum(x^i)^2\}^{1/2}. \tag{1.1}$$

To make sense of (1.1) one selects some asymptotically Minkowskian coordinates $\{x^\mu\}$ in which the metric $\mathfrak{g}$ takes the form[1]

$$\mathfrak{g}(\partial/\partial x^\mu, \partial/\partial x^\nu) = g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, h_{\mu\nu}(x) = \mathcal{O}_1(r(x)^{-\alpha}), \tag{1.2}$$

* On leave of absence from the Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland

** Supported in part by the Polish Ministry of Science Research Project CPBP 01.2 and by the NSF Grant PHY8503072 to Yale University

[1] The signature is $+2$, greek indices run from 0 to 4, latin ones from 1 to 3, $\eta_{\mu\nu}$ is the flat Minkowski metric, $dS_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}dx^\alpha \wedge dx^\beta/2$, $\varepsilon_{0123} = 1$. We shall write $f = \mathcal{O}_n(r^\alpha)$, $\alpha\in\mathbb{R}$, if f satisfies $|f| \leqq C\sigma^\alpha$, $|\nabla_\mu f| \leqq C\sigma^{\alpha-1}, |\nabla_{\mu_1}\cdots\nabla_{\mu_n}f| \leqq C\sigma^{\alpha-n}$, with $\sigma = (1+r^2)^{1/2}$, for some constant $C, O(r^\alpha) = \mathcal{O}_0(r^\alpha), f = o(r^\alpha)$ if $\lim_{r\to\infty} r^{-\alpha}f = 0$, r^0 is always understood as $\ln r$. $B(R)$ and $S(R)$ denote a coordinate ball and sphere of radius R respectively (if ambiguities are likely to occur the coordinate sphere in e.g. coordinates y will be denoted by $S_y(R)$, etc.). Letters C, C', etc. are used throughout to denote strictly positive constants which may vary from line to line