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The Stability and Instability of Relativistic Matter

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Abstract. We consider the quantum mechanical many-body problem of electrons and fixed nuclei interacting via Coulomb forces, but with a relativistic form for the kinetic energy, namely $p^2/2m$ is replaced by $(p^2c^2+m^2c^4)^{1/2}-mc^2$. The electrons are allowed to have q spin states ($q=2$ in nature). For one electron and one nucleus instability occurs if $z\alpha>2/\pi$, where z is the nuclear charge and α is the fine structure constant. We prove that stability occurs in the many-body case if $z\alpha\leqq 2/\pi$ and $\alpha<1/(47q)$. For small z, a better bound on α is also given. In the other direction we show that there is a critical α_c (no greater than $128/15\pi$) such that if $\alpha>\alpha_c$ then instability always occurs for *all* positive z (not necessarily integral) when the number of nuclei is large enough. Several other results of a technical nature are also given such as localization estimates and bounds for the relativistic kinetic energy.

I. Introduction

One of the early important successes of quantum mechanics was the interpretation of the stability of the hydrogen atom. The ground state energy of the hydrogen Hamiltonian is finite and thus the hydrogen atom is stable quantum mechanically, even though it is unstable classically. The Coulomb singularity $-ze^2/r$ is controlled by a new feature of Schrödinger mechanics, the uncertainty principle. While the stability of the hydrogen atom is clear and simple, a more subtle question arises when many particles are taken into account. It is convenient to distinguish two notions of stability.

Stability of the first kind: The ground state energy is finite.

Stability of the second kind: The ground state energy is bounded below by a constant times the number of particles.

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