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Hyper-Kähler Metrics and a Generalization of the Bogomolny Equations

Henrik Pedersen[1] and Yat Sun Poon[2]

[1] Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK-5230 Odense M, Denmark

[2] Department of Mathematics, Rice University, Houston, TX 77251, USA

Abstract. We generalize the Bogomolny equations to field equations on $\mathbb{R}^3 \otimes \mathbb{R}^n$ and describe a twistor correspondence. We consider a general hyper-Kähler metric in dimension $4n$ with an action of the torus T^n compatible with the hyper-Kähler structure. We prove that such a metric can be described in terms of the T^n-solution of the field equations coming from the twistor space of the metric.

1. Introduction

Let $\tilde{E}$ be a rank k complex vector bundle on $\mathbb{R}^3 \otimes \mathbb{R}^n$ with a connection ∇ and n sections of the adjoint bundle $\Phi^1, \dots, \Phi^n$, the Higgs fields. Let x_α^i, $i = 1, \dots, n$, $\alpha = 1, 2, 3$ be the coordinates on $\mathbb{R}^3 \otimes \mathbb{R}^n$ and consider the field equations

$$\left.\begin{aligned} F_{x_\alpha^i x_\beta^j} &= \sum_\gamma \varepsilon_{\alpha\beta\gamma} \nabla_{x_\gamma^i} \Phi^j + \tfrac{1}{2}\delta_{\alpha\beta}[\Phi^i, \Phi^j] \\ \nabla_{x_\alpha^i} \Phi^j &= \nabla_{x_\alpha^j} \Phi^i \end{aligned}\right\}, \tag{1.1}$$

where $F = \sum F_{x_\alpha^i x_\beta^j} dx_\alpha^i \wedge dx_\beta^j$ is the curvature.

In each $\mathbb{R}^3$ obtained by fixing a vector in the $\mathbb{R}^n$ factor of $\mathbb{R}^3 \otimes \mathbb{R}^n$ the field equations reduce to the *Bogomolny equations* by contracting the fields with the vector, [5]. This is the generalization mentioned in the title. We prove that there is a twistor correspondence between solutions to these equations and holomorphic rank k bundles on $T = \mathcal{O}(2) \otimes \mathbb{C}^n$ trivial on real sections of $T \to \mathbb{CP}^1$.

We shall consider the field equations for the abelian torus T^n and their relation to *hyper-Kähler* geometry: Let (M, g) be a $4n$-dimensional Riemannian manifold with three almost complex structures I, J and K satisfying the quaternion algebra identities

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K$$

etc. Assume that g is Hermitian with respect to I, J and K, i.e.

$$g(IX, IY) = g(X, Y), \quad X, Y \in TM$$