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Approximate Neutrality of Large-Z Ions*

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Abstract. Let $N(Z)$ denote the number of electrons which a nucleus of charge Z can bind in non-relativistic quantum mechanics (assuming that electrons are fermions). We prove that $N(Z)/Z \to 1$ as $Z \to \infty$.

1. Introduction

This paper is a contribution to the exact study of Coulombic binding energies in quantum mechanics. Let $H(N, Z)$ denote the Hamiltonian

$$H(N,Z) = \sum_{i=1}^{N} (-\Delta_i - Z|x_i|^{-1}) + \sum_{i<j} |x_i - x_j|^{-1},$$

and let $E(N, Z)$ denote its minimum over all fermion states (we suppose there are two spin states allowed, although any fixed number could be accommodated). For comparison purpose, we let $E_b(N, Z)$ denote the same minimum, but over all states (taken on a totally symmetric wave function, hence b for boson).

It is a fundamental result of Ruskai [9] for bosons, and Sigal [11] for fermions (see also Ruskai [10]) that there exists $N(Z), N_b(Z)$ so that, for all $j = 0, 1, \ldots,$

$$E(N(Z), Z) = E(N(Z) + j, Z); \quad E_b(N_b(Z), Z) = E_b(N_b(Z) + j, Z).$$

We let $N(Z)$ (respectively $N_b(Z)$) denote the smallest number for which the first (respectively second) equality holds for all j. Sigal [12] showed that

$$\overline{\lim}\,[N(Z)/Z] \leqq 2, \quad \lim\,[\ln N_b(Z)/\ln Z] \leqq 1, \tag{1.1}$$

and then Lieb [6, 7] proved the bounds

$$N(Z) < 2Z + 1, \quad N_b(Z) < 2Z + 1 \tag{1.2}$$

which implies, in particular, that a doubly ionized hydrogen atom is unstable.

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