Determinants, Torsion, and Strings

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Abstract. We apply the results of [BF1, BF2] on determinants of Dirac operators to String Theory. For the bosonic string we recover the "holomorphic factorization" of Belavin and Knizhik. Witten's global anomaly formula is used to give sufficient conditions for anomaly cancellation in the heterotic string (for arbitrary background spacetimes). To prove the latter result we develop certain torsion invariants related to characteristic classes of vector bundles and to index theory.

String Theory has spawned a vigorous interaction between mathematics and physics. This intermingling of two quite separate intuitions is fruitful for both disciplines. Of particular value for mathematicians are the concrete examples generated and discussed by physicists. The purpose of this paper is to consider the relationship of some examples to the circle of ideas surrounding the Atiyah-Singer Index Theorem.

Atiyah and Singer first demonstrated the connection between determinants (in Quantum Field Theory) and the families index theorem [AS1]. Their concern was with anomalies, which they interpreted as nontrivial topology (over the reals) in the determinant line bundle $\mathscr{L}$. Witten's work on global anomalies [W1] suggested a more refined geometric picture: $\mathscr{L}$ has a natural connection whose curvature and holonomy represent the local and global anomaly, respectively.[2] The mathematical ideas used to construct such a connection are largely due to Quillen [Q1] who, for entirely different reasons, introduced a metric on $\mathscr{L}$. The connection on $\mathscr{L}$ was rigorously constructed in [BF1]. The analytical techniques developed by Bismut in [B] were used in [BF2] to derive the formulae for its curvature and holonomy. We discuss these developments in Sect. 1.

In Sect. 2 we interpret some known results about the bosonic string in terms of the geometry of the determinant line bundle. We emphasize that purely topological methods do not suffice here. In fact, our main purpose in this paper is

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[2] Witten explicitly stated that his work could be interpreted in terms of this connection