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Renormalization Group Flow for General σ-Models

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Abstract. The renormalization group flow for σ-models with base space of dimension 1 or 2 is investigated. In two dimensions it is shown that the flow is singular towards the UV for a generic target space. In one dimension it is shown that there are IR fixed points coming from negatively curved symmetric spaces.

The quantum σ-model based on a Riemannian manifold M is the quantum field theory of harmonic maps from $\mathbb{R}^d$ to M. This is a perturbatively renormalizable theory (in dim. reg.) when $d \leqq 2$. For $d=2$ the classical theory has a scale invariance which is generally broken on the quantum level, the breaking being given by the β-function. This function was first computed to the two-loop level in [1], which also studied the flow toward the infra-red. Such is the region of interest in statistical mechanics, in which IR fixed points give the long distance limits of scalar theories for $d<4$ [2]. In other areas, the tree level of (super) string theories is given by (super) quantum σ-models. There is evidence that for various degrees of supersymmetry the β-function is given exactly by its one-loop approximation, which would imply that in order to have the scale invariance desired for string theory, it suffices that M be a ((hyper)-Kähler) Ricci-flat manifold [3]. Finally, the Hamiltonian of the quantum σ-model is a renormalized version of the formal Laplacian + potential term acting on functions on the loop space ΩM. This may be of mathematical interest.

It is an open question as to when the quantum σ-model exists as a continuum field theory. So far it has been constructed in the $d=2$ case when $M=S^N$ and a hierarchical propagator is used [4]. It is generally believed that for a continuum QFT based on perturbation theory to exist, one must have asymptotic freedom at short distance. This is needed so that one can solve the renormalization group (RG) flow without singularities toward the UV. For the $d=2$ σ-model, the asymptotic freedom condition means that M must be Ricci-nonnegative. We wish to look more carefully at the flow toward the UV. We find that asymptotic freedom is not enough. One also needs an exceptional smoothness of the metric which