Commun. Math. Phys. 107, 1–20 (1986)

Communications in Mathematical Physics

The Stability of Rotating Vortex Patches*

Yieh-Hei Wan**
Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14214, USA

Abstract. In this paper we examine the nonlinear and linear stability of various *rotating* vortex patches. These patches include the Kirchhoff ellipse, the Kelvin waves, and the co-rotating uniform m vortices. These are achieved by using relative variational methods and spectral analysis. Thus, we extend Arnol'd's idea for stability problems in [1965, 1969] to a non-smooth symmetric setting and also relate that to the usual linear stability analysis.

1. Introduction

Consider the motion of an incompressible flow with unit density in R^2 in the absence of external forces. At any instant, the velocity field $(u,v)=(\psi_{\bar{y}}, -\psi_{\bar{x}})$ for some stream function ψ on $R^2=\{\mathbf{x}=(\bar{x},\bar{y})\}$. The *vorticity* $\omega=v_{\bar{x}}-u_{\bar{y}}=-\psi_{\bar{x}\bar{x}}-\psi_{\bar{y}\bar{y}}=-\Delta\psi$. We like to use the vorticity ω as the independent variable. Given ω, let us choose a stream function $\psi=\int G\omega=(1/2\pi)\int_{R^2}\omega(\bar{\mathbf{x}}')\ln(1/|\bar{\mathbf{x}}'-\bar{\mathbf{x}}|)d\mathbf{x}'$, so that the velocity field is zero at infinity. The vorticity evolves according to the vorticity equation: $\omega_t+u\omega_{\bar{x}}+v\omega_{\bar{y}}=0$. Denote by $\Phi_t(\omega)$ the vorticity at time t, with initial vorticity ω.

The *energy* E, the *circulation* C, the *centre* $(\bar{x}_0,\bar{y}_0)$, and the *angular momentum* J are preserved under the motion Φ_t. Recall that for a given vorticity ω, $E=\frac{1}{2}\langle\omega,\psi\rangle=\frac{1}{2}\int_{R^2}\omega(\bar{\mathbf{x}})\psi(\bar{\mathbf{x}})d\bar{\mathbf{x}}$, $C=\int_{R^2}\omega(\bar{\mathbf{x}})d\bar{\mathbf{x}}$, $\bar{x}_0=\int_{R^2}\bar{x}\omega(\bar{\mathbf{x}})d\bar{\mathbf{x}}$, $\bar{y}_0=\int_{R^2}\bar{y}\omega(\bar{\mathbf{x}})d\bar{\mathbf{x}}$, and $J=\int_{R^2}|\bar{\mathbf{x}}|^2\omega(\bar{\mathbf{x}})d\bar{\mathbf{x}}$. *A vortex patch* ω is a vorticity in the form χ_A, where χ_A stands for the characteristic function for a bound (measurable) set A in R^2. χ_{A_j} is called a component of χ_A, if A_j is a component of A. Vortex patches and their components are all preserved under the motion Φ_t.

A vortex patch χ_A is said to be *stationary* if $\Phi_t(\chi_A)=\chi_A$ for all $t\geqq 0$. A vortex patch is said to be *rotating* if $\Phi_t(\chi_A)=\chi_{R_{\Omega t}A}$ for all $t\geqq 0$, where R_θ stands for a rotation through angle θ. The Kirchhoff vortices χ_E (see Sect. 4) in which E is an ellipse, are our model for rotating vortex patches. Two families of rotating vortex patches have been found recently. They are (a) the m-fold symmetric "Kelvin" waves

** Supported by National Science Foundation under grant DMS-8501746
* Dedicated to the memory of my grandmother, Lang-Chang Lee Wan