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BV Estimates Fail for Most Quasilinear Hyperbolic Systems in Dimensions Greater Than One

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Abstract. We show that for most non-scalar systems of conservation laws in dimension greater than one, one does not have BV estimates of the form

$$\|\nabla u(\bar{t})\|_{\mathrm{T.V.}} \leqq F(\|\nabla u(0)\|_{\mathrm{T.V.}}),$$
$$F \in C(\mathbb{R}), \quad F(0)=0, \quad F \text{ Lipshitzean at } 0,$$

even for smooth solutions close to constants. Analogous estimates for L^p norms

$$\|u(\bar{t})-\bar{u}\|_{L^p} \leqq F(\|u(0)-\bar{u}\|_{L^p}), \qquad p \neq 2$$

with F as above are also false. In one dimension such estimates are the backbone of the existing theory.

The assertions of the abstract are fairly direct consequences of the fact that, except for trivial cases, linear hyperbolic systems in dimension greater than one are not well posed in L^p for $p \neq 2$. One might hope that the conservation laws are better behaved than the linear systems. For example, in one space dimension the solution operator $u(0) \to u(t)$, $t>0$, maps L^∞ to BV, a smoothing property not shared by linear equations. It is the purpose of this note to dash such hopes. The analysis is made entirely within the framework of smooth solutions so that neither conservation form nor entropy conditions play a role.

The $k \times k$ systems for $u(t, x)$, $t, x \in \mathbb{R} \times \mathbb{R}^d$, $u \in \mathbb{R}^k$ are assumed to have the form

$$A_0(u)\partial_t u + \sum_{j=1}^{d} A_j(u)\partial_j u + B(u) = 0, \tag{1}$$

where $\partial_j \equiv \partial/\partial x_j$ and

$$A_j \in C^\infty(\mathbb{R}^k : \mathrm{Hom}(\mathbb{R}^k)), \qquad B \in C^\infty(\mathbb{R}^k : \mathbb{R}^k).$$

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