Commun. Math. Phys. 106, 395–426 (1986)

Communications in Mathematical Physics

New Proofs of the Existence of the Feigenbaum Functions

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Abstract. A new proof of the existence of analytic, unimodal solutions of the Cvitanović-Feigenbaum functional equation $\lambda g(x) = -g(g(-\lambda x))$, $g(x) \approx 1 - \mathrm{const}|x|^r$ at 0, valid for all λ in $(0, 1)$, is given, and the existence of the Eckmann-Wittwer functions [8] is recovered. The method also provides the existence of solutions for certain given values of r, and in particular, for $r=2$, a proof requiring no computer.

0. Notations

If $z \in \mathbf{C}$, we denote z^* its complex conjugate, and reserve the notation $\bar{S}$ to denote the closure of a set S.

Let J be an open, possibly empty interval in $\mathbf{R}$. We denote

$$\mathbf{C}(J) = \{z \in \mathbf{C}\colon \operatorname{Im} z \neq 0 \text{ or } z \in J\}.$$

In particular, $\mathbf{C}(\emptyset) = \mathbf{C}_+ \cup \mathbf{C}_-$, where

$$\mathbf{C}_+ = -\mathbf{C}_- = \{z \in \mathbf{C}\colon \operatorname{Im} z > 0\}.$$

$\mathbf{F}(J)$ is the real Fréchet space of functions f, holomorphic on $\mathbf{C}(J)$, with $f(z^*)^* = f(z)$, equipped with the topology of uniform convergence on compact subsets of $\mathbf{C}(J)$. $\mathbf{P}(J)$ is the subset of $\mathbf{F}(J)$ consisting of the functions f such that $f(\mathbf{C}_+) \subset \bar{\mathbf{C}}_+$, and $f(\mathbf{C}_-) \subset \bar{\mathbf{C}}_-$. These functions are often called Herglotz or Pick functions.

$\mathbf{P}_0(J)$ is the subset of $\mathbf{P}(J)$ consisting of the functions f such that $|f(z)/z| \to 0$ as $z \to \infty$ in non-real directions.

1. Introduction

The functional equation

$$g(x) = -\frac{1}{\lambda} g(g(-\lambda x)) \tag{1.1}$$