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Inequalities for the Schatten p-Norm. III

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Abstract. We present some inequalities for the Schatten p-norm of operators on a Hilbert space. It is shown, among other things, that if A is an operator such that $\operatorname{Re} A \geqq a \geqq 0$, then for any operator X, $\|AX+XA^*\|_p \geqq 2a\|X\|_p$. Also, for any two operators A and B, $\||A|-|B|\|_2^2 + \||A^*|-|B^*|\|_2^2 \leqq 2\|A-B\|_2^2$.

In their investigation on the quasi-equivalence of quasi-free states of canonical commutation relations, Araki and Yamagami [1] proved that for any two bounded linear operators A and B on a Hilbert space H, $\||A|-|B|\|_2 \leqq 2^{1/2}\|A-B\|_2$. Also, in working on the approach to an equilibrium in harmonic chain or the elementary excitation spectrum of a random ferromagnet, as mentioned in [3], one may encounter the following useful inequality due to van Hemmen and Ando [3, Lemma 3.1]. If X is a compact operator and A is an operator such that $A \geqq a \geqq 0$, then $\|AX+XA\|_p \geqq 2a\|X\|_p$. This inequality is related to the one proved by the author in [4, Theorem 3]. The inequality in [4, Theorem 3] is equivalent to that $\|AX+XA^*\|_p \geqq a\|X\|_p$ for any operator X, whenever $\dfrac{A+A^*}{2} \geqq a \geqq 0$. But as seen from the proof if X is assumed to be self-adjoint (or even seminormal), then $\|AX+XA^*\|_p \geqq 2a\|X\|_p$.

It is the object of this note to present the best possible extension of this result by removing the restriction on X. We will prove a general theorem which gives the above mentioned inequalities in [3 and 4] as corollaries. The technique developed for this purpose proves to be useful also in extending the Araki and Yamagami result and it is likely to have further applications.

An operator means a bounded linear operator on a separable, complex Hilbert space H. Let $B(H)$ denote the algebra of all bounded linear operators acting on H. Let $K(H)$ denote the ideal of compact operators on H. For any compact operator A, let $s_1(A), s_2(A), \ldots$ be the eigenvalues of $|A|=(A^*A)^{1/2}$ in decreasing order and repeated according to multiplicity. A compact operator A is said to be in the Schatten p-class C_p $(1 \leqq p < \infty)$, if $\sum_i s_i(A)^p < \infty$. The Schatten p-norm of A is