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2- and 3-Cochains in 4-Dimensional SU(2) Gauge Theory

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Abstract. Explicit formulae are derived for the 2- and 3-cochains $\Omega^{(2)}_{\mu\nu\varrho}(i,j,k)$ and $\Omega^{(3)}_{\mu\nu\varrho\sigma}(i,j,k,\ell)$ in SU(2) gauge theory in 4 dimensions. It turns out that $\Omega^{(3)}_{\mu\nu\varrho\sigma}(i,j,k,\ell)$ is given by the volume of a spherical tetrahedron spanned by the gauge transformations relating the gauges i,j,k,l.

I. Introduction

Higher-order cocycles

$$\omega^{(n)} = \int \alpha^{3-n}\sigma_{\mu\ldots}\Omega^{(n)}_{\mu\ldots} \tag{1}$$

(here written for 4 space-time dimensions), where $\Omega^{(n)}_{\mu\ldots}$ is the n-cochain, play an important role in group representation theory, in the investigation of the structure of anomalies, Wess-Zumino effective actions and groups associated with a Kac-Moody algebra [1] as well as in the derivation of a closed expression for the topological charge [2]. It is therefore of great interest to know $\Omega^{(n)}_{\mu\ldots}$ explicitly. In this paper we shall consider the case of gauge group SU(2) in 4 dimensions and derive explicit expressions for $\Omega^{(2)}_{\mu\nu\varrho}$ and $\Omega^{(3)}_{\mu\nu\varrho\sigma}$.

The starting-point is the Chern-Pontryagin density

$$P = -\frac{1}{32\pi^2}\varepsilon_{\mu\nu\varrho\sigma}\operatorname{Tr}[F^i_{\mu\nu}F^i_{\varrho\sigma}],\; F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + [A^i_\mu, A^i_\nu], \tag{2}$$

where the index i specifies a particular gauge. The 4-dimensional integral of P is the topological charge, which is an invariant. The Chern-Pontryagin density can be written as a total divergence,

$$P = \partial_\mu \Omega^{(0)}_\mu(i), \tag{3}$$

where

$$\Omega^{(0)}_\mu(i) = -\frac{1}{8\pi^2}\varepsilon_{\mu\nu\varrho\sigma}\operatorname{Tr}[A^i_\nu(\partial_\varrho A^i_\sigma + \tfrac{2}{3}A^i_\varrho A^i_\sigma] \tag{4}$$