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A Shorter Proof of the Existence of the Feigenbaum Fixed Point

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Abstract. We use the Leray-Schauder Fixed Point Theorem to prove the existence of an analytic fixed point for the period doubling accumulation renormalization operator. Our argument does not, however, show that the linearization of the renormalization operator at this fixed point is hyperbolic.

1. Introduction

Two independent proofs (Campanino et al. [1, 2], Lanford [4]) have been given for the existence of an even analytic solution to the Feigenbaum-Cvitanović functional equation [3]

$$g(x) = -\frac{1}{\lambda} g(g(-\lambda x)), \qquad g(0) = 1 \tag{1.1}$$

with

$$g''(0) < 0; \; \lambda \equiv -g(1) > 0 .$$

Both of these proofs rely on extensive computations. In this paper, we give yet another proof, based on the Leray-Schauder Fixed Point Theorem, which, if still fundamentally computational in nature, requires a substantially smaller amount of computation. It should be noted that the argument given here, like that of Campanino et al. and unlike the author's computer assisted proof, does *not* establish the spectral properties of the linearization of the renormalization operator at the fixed point g which are essential for the application of g to the analysis of period-doubling accumulation.

We will work in a space of even mappings f of $[-1, 1]$ to itself, satisfying the normalization condition

$$f(0) = 1 , \tag{1.2}$$

expressed as functions of x^2. Since we are working with x^2 as the independent variable, the renormalization operator has the form

$$\mathcal{T}f(x) = -\frac{1}{\lambda} f([f(\lambda^2 x)]^2), \qquad \lambda \equiv -f(1) . \tag{1.3}$$