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Differential Geometric Statement of Variational Equations for Abstract Fluids

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Abstract. A new global approach for the variational equations of fluids is given. The concept of prefluid is introduced, together with its variational equations and examples.

1. Introduction

The intrinsic description of perfect fluids in Newtonian space-time that appears mostly in the literature (see [1, 3]) resembles the one used for elastic media: the study of a one-parameter (the time) family of diffeomorphisms. Besides its kinematic flavor, the limitation of this approach is evident for example in the case of general relativity, where there are no preferred spacelike slices for the parametrization of those diffeomorphisms.

For compressible fluids the situation is even worse, for in addition to the above relativistic remark, we are confronted with the heterogeneous role of the equation of continuity, which is to be imposed as a constraint. Also there are two classical pictures, Euler's and Lagrange's, with corresponding variational principles (variations on the velocities, variations on the initial positions, see [7] for discussion), that cast more confusion into the differential geometric core of the problem.

To avoid these shortcomings we introduce two innovations. The first one is to take fluids as what they seem to be, i.e. vector densities. The second refers to the variational equations and consists of taking variations that proceed from vector fields on the base space instead of vector fields tangent to the fibres.

Thus, let $\pi : E^{(r)} \to M^n$ be the bundle of r-vector densities on M, where M could be a configuration space-time. Let $\pi_\infty : J^\infty(\pi) \to M$ be the bundle of ∞-jets of local cross-sections of π. Then an abstract fluid is a section of π. If a π_∞-horizontal n-form λ (the Lagrangian) is given on $J^\infty(\pi)$, it defines a functional on abstract fluids in the standard way. If X is a vector field on M with flow ϕ_t, it induces a vector field on the bundle of frames $F(M)$ via the flow ϕ_{t*}. This new vector field induces a vector field in each associated bundle, and in particular, a vector field $\tilde{X}$ on $E^{(r)}$ that induces the desired variation on abstract fluids.

We show that the variational principle that corresponds to such variations leads in the case of conservative fluids to Euler's equation *plus* the equation of continuity. Therefore, the latter becomes a consequence of the variational