Commun. Math. Phys. 94, 271–287 (1984)

Communications in Mathematical Physics

The Algebraic Complete Integrability of Geodesic Flow on $SO(N)$

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Abstract. We study for which left invariant diagonal metrics λ on $SO(N)$, the Euler-Arnold equations

$$\dot{X}=[X,\lambda(X)],\ X=(x_{ij})\in so(N),\ \lambda(X)_{ij}=\lambda_{ij}x_{ij},\ \lambda_{ij}=\lambda_{ji}$$

can be linearized on an abelian variety, i.e. are solvable by quadratures. We show that, merely by requiring that the solutions of the differential equations be single-valued functions of complex time $t\in\mathbb{C}$, suffices to prove that (under a non-degeneracy assumption on the metric λ) the only such metrics are those which satisfy Manakov's conditions $\lambda_{ij}=(b_i-b_j)\ (a_i-a_j)^{-1}$. The case of degenerate metrics is also analyzed. For $N=4$, this provides a new and simpler proof of a result of Adler and van Moerbeke [3].

Introduction

Recently the question of understanding the complete integrability (or the non-integrability) of a Hamiltonian system has regained considerable interest. For example, Adler and van Moerbeke [2, 3] have discussed and used a criterion to decide what they propose to call the algebraic complete integrability of a Hamiltonian system and, in a completely different vein, Ziglin [16, 17] has proved the (global) non-analytic integrability of the motion of a rigid body around a fixed point in the presence of gravity except in the three famous well known cases of Euler, Lagrange and Kowalewski (see also Holmes and Marsden [11]). One of the most fascinating common features of all these investigations is the connection between the question of the complete integrability of a Hamiltonian system and the behaviour of its solutions as functions of *complex* time. Up to now, this connection is not understood in general. In this note, we provide an interesting new example of this connection by showing that, merely by requiring that the solutions of the differential equations be single-valued functions of $t\in\mathbb{C}$, suffices to

* Aspirant Fonds National Belge de la Recherche Scientifique (F.N.R.S.)
Supported in part by N.S.F. Grant 8102696 while visiting Brandeis University