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A New Combinatoric Estimate for Cluster Expansions*

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Abstract. We state and prove a new and previously unsuspected tree graph inequality, which is significantly stronger than the one commonly applied to cluster expansions. The older inequality controls the counting problem in the convergence proof of such an expansion, but the new inequality does more: it also exhibits extra $1/n!$ factors that can be applied to the cancellation of number divergences. The proof of this new combinatoric estimate is completely elementary.

1. Introduction

For the past two years a new method of tree graph estimation has been applied to cluster expansions in statistical mechanics and quantum field theory [1–4]. The technique is based on a very powerful and flexible combinatoric identity that first appeared in [1]. However, the method has been slow to attract attention – in part, because the combinatoric identity has to be supplemented with additional topological ordering concepts if one wishes to apply it to the problem of cancelling the number divergence appearing in a cluster expansion. The goal of this paper is to prove a new and different combinatoric estimate whose application is equally powerful but does not require these auxiliary notions.

For convenience we repeat the basic definitions given in [1].

Definition 1.1. Let (S, p_1) be a pointed set with cardinality N. An *ordered connectivity graph* (o.c.g.) *on* (S, p_1) is a mapping G from $\{2, \ldots, N\}$ into $S \times S$ such that if $G(i) = (g_1(i), g_2(i))$, then

(a) $g_2(i) \neq g_2(j)$ for $i \neq j$,

(b) $g_1(i) \in \{p_1\} \cup \{g_2(j) \mid j < i\}$.

Definition 1.2. A *tree graph of order* N is a mapping η from $\{2, \ldots, N\}$ into $\{1, \ldots, N-1\}$ such that $\eta(i) < i$.

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