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Vacuum Charge and the Eta Function

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Abstract. The vacuum charge of a second quantized spinor field in a static classical background field on a static spacetime is studied. When $g_{00}=1$ the vacuum charge is shown to be essentially the eta function of the spinor Hamiltonian at $s=0$. This is computed for compact and noncompact spaces and a boundary dependence is derived in the latter case.

1. Introduction

A phenomenon of current interest in quantum field theory is the possible nonzero electrical charge of the vacuum state. Jackiw and Rebbi showed that when a fermion field is second-quantized in the background of a soliton the ground state can be degenerate with the different vacua having half-integral charge [1]. Later, Goldstone and Wilczek demonstrated that any vacuum charge could be obtained if the conjugation symmetry of the fermion in the external field is abandoned [2]. Their method of calculation was to sum the Feynman diagrams for the expectation value of the current of a free fermion in a slowly-varying background field. We shall show how the vacuum charge can be computed nonperturbatively in the most general static case by identifying it with the eta function of spectral geometry.

For an elliptic self-adjoint pseudo-differential operator H acting on cross-sections of a vector bundle over a compact manifold M without boundary, the eta function is defined as $\eta_H(s)=\sum_{\lambda_i\neq 0}\lambda_i|\lambda_i|^{-s-1}$, where the sum is over the nonzero eigenvalues of H [3]. Although this series only converges for $\operatorname{Re}s>\dim M/\operatorname{order}H$, it can be analytically continued to the whole s-plane. Its value $\eta_H(0)$ is a regularized measure of the spectral asymmetry of H. Remarkably, η_H is always holomorphic at $s=0$ [3, 4]. This value $\eta_H(0)$ enters in the integral formulae for characteristic classes of a manifold with boundary [3].

We shall show that on a compact manifold the vacuum charge essentially is the eta function. We also show that a natural extension of the eta function to noncompact manifolds gives the boundary dependence of the vacuum charge found in [2], but generalized to the case of an arbitrary static Hamiltonian.

2. Vacuum Charge in a Static Spacetime

Notation. Greek letters will denote four-dimensional indices and latin letters will denote three-dimensional indices. We let $\sqrt{g}$ and $\sqrt{g^{(3)}}$ denote the four- and three-