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Interpolation Theory and the Wigner–Yanase–Dyson–Lieb Concavity

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Abstract. The Wigner–Yanase–Dyson–Lieb concavity is naturally captured in the frame of interpolation theory. Among other results, a certain generalization (involving operator monotone functions) of this concavity in the context of general von Neumann algebras is obtained. Also, a close relationship between the above subjects and F. Hansen's inequality is clarified. All results are proved by using simple variational expressions of involved quantities.

0. Introduction

Since the joint concavity of $\mathrm{Tr}(a^{1-\theta}x^*b^\theta x)$, $0 \leqq \theta \leqq 1$, in (a, b) was proved by Lieb, [22], it is referred to as the Wigner–Yanase–Dyson–Lieb (WYDL) concavity. Here, a and b (respectively x) are positive (respectively an arbitrary) matrices. Then, Araki, [6], obtained the corresponding result for general von Neumann algebras. Due to the fact that this concavity has important applications to theoretical physics and information theory (subadditivity for entropy, etc., see Sect. 8, [28], for example), several authors have been trying to obtain various generalizations in many directions, [11, 24, 25, 30].

Proofs in [6, 22] are based on the Phragmen–Lindelöf theorem (complex interpolation). The purpose of the article is to capture the WYDL concavity in the frame of general interpolation theory. Especially, we examine a certain real interpolation method (the K-method of Peetre) in Sect. 1, and quadratic interpolation methods in Sects. 2 and 3. Also, as applications of our arguments, in Sect. 4 we establish a close relation among this subject, F. Hansen's inequality [15], and operator monotone functions [10], while in Sect. 5 we obtain a certain generalization of Araki's version, [6], of the WYDL concavity in the context of general von Neumann algebras.

We show that, in our frame, the WYDL concavity is a natural and common phenomenon (Theorems 1.8, 3.5) due to the fact that many involved quantities

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